SECTION #1.

SOLVING LINEAR SYSTEMS. ECHELON FORM.

1) SOLUTIONS of A LINEAR SYSTEM.

$$(*)\quad \begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & & a_{mn} \end{pmatrix} \quad \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \quad \text{GIVEN}.$$

m - EQUATIONS (m - ROWS)

n - UNKNOWNS (VARIABLES) (n - COLUMNS.

Possibilities. ① NO SOLUTION ((*) IS INCONSISTENT)

② ONE SOLUTION } (*) CONSISTENT

③ INFINITELY MANY SOLUTIONS } (*) CONSISTENT

EX 1

$$\begin{cases} 2x_1 + 3x_2 - x_3 = 4 \\ -x_1 - x_2 + x_3 = -2 \\ x_2 + x_3 = 1 \\ 4x_1 - 2x_2 + 6x_3 = 0. \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} 2 & 3 & -1 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \\ 4 & -2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ -1 & -1 & 1 & -2 \\ 0 & 1 & 1 & 1 \\ 4 & -2 & 6 & 0 \end{array}\right)$$ ARGUMENTED MATRIX.

MULTIPLY THE FIRST EQ by $1/2$ AND ADD IT TO THE 2nd EQ.

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 1 & 1 \\ 4 & -2 & 6 & 0 \end{array}\right) \leftrightarrow R_2 \leftarrow R_2 + \frac{1}{2} R_1$$

MULTIPLY THE 2nd EQ by -2 AND ADD IT TO THE 3rd EQ

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 0 & 1/2 & 1/2 & 2 \\ 0 & 0 & 0 & 1 \\ 4 & -2 & 6 & 0 \end{array}\right) \leftrightarrow R_3 \leftarrow R_3 - 2R_2$$

NO SOLUTION

$$\boxed{0x_1 + 0x_2 + 0x_3 = 1}$$

EX 2

$$\begin{cases} 2x_1 + 3x_2 - x_3 = 4 \\ -x_1 - x_2 + x_3 = -2 \\ x_2 + x_3 = 0 \\ 4x_1 - 2x_2 + 4x_3 = 0 \end{cases}$$

$$\begin{pmatrix} 2 & 3 & -1 & 4 \\ -1 & -1 & 1 & -2 \\ 0 & 1 & 1 & 0 \\ 4 & -2 & 4 & 0 \end{pmatrix}$$

$R_2 \leftarrow R_2 + \frac{1}{2}R_1$

$R_4 \leftarrow R_4 - 2R_1$

$$\begin{pmatrix} 2 & 3 & -1 & 4 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -8 & 6 & -8 \end{pmatrix}$$

$R_3 \leftarrow R_3 - 2R_2$

$R_4 \leftarrow R_4 + 16R_2$

$$\begin{pmatrix} \underline{2} & 3 & -1 & 4 \\ 0 & \underline{\frac{1}{2}} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 14 & -8 \end{pmatrix}$$

$R_3 \leftrightarrow R_4$

LEADING COEFF'S

$$\begin{pmatrix} \underline{2} & 3 & -1 & 4 \\ 0 & \underline{\frac{1}{2}} & \frac{1}{2} & 0 \\ 0 & 0 & \underline{14} & -8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

ECHELEN FORM

$R_3 \leftarrow R_3 \cdot \frac{1}{14}$

$R_2 \leftarrow R_2 \cdot 2$

$R_1 \leftarrow R_1 \cdot \frac{1}{2}$

$$\begin{pmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{4}{7} \end{pmatrix}$$

$R_1 \leftarrow R_1 + \frac{1}{2} R_3$

$R_2 \leftarrow R_2 - R_3$

$$\begin{pmatrix} 1 & 3/2 & 0 & 12/7 \\ 0 & 1 & 0 & 4/7 \\ 0 & 0 & 1 & -4/7 \end{pmatrix}$$

$R_1 \leftarrow R_1 - \frac{3}{2} R_2$

$$\begin{pmatrix} 1 & 0 & 0 & 6/7 \\ 0 & 1 & 0 & 4/7 \\ 0 & 0 & 1 & -4/7 \end{pmatrix}$$

SOLUTION (UNIQUE)

$x_1 = 6/7$

$x_2 = 4/7$

$x_3 = -4/7$.

CHECK IT.

EX 3

$$\begin{cases} 2x_1 + 3x_2 - x_3 = 4 \\ -x_1 - x_2 + x_3 = -2 \\ x_2 + x_3 = 0 \\ x_1 + 3x_2 + x_3 = 2 \end{cases}.$$

$$\begin{pmatrix} 2 & 3 & -1 & 4 \\ -1 & -1 & 1 & -2 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 3 & -1 & 4 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3/2 & 3/2 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 2 & 3 & -1 & 4 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3/2 & -1/2 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(ECHELED FORM)

COLUMN WITHOUT LEADING COEFF.

$$\begin{cases} x_1 + \frac{3}{2}x_2 - \frac{x_3}{2} = 2 \\ x_2 + x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 + \frac{3}{2}x_2 = 2 - \frac{x_3}{2} \\ x_2 = -x_3 \end{cases}$$

$\Rightarrow$ $\boxed{x_2 = -x_3}$

$$x_1 = -\frac{3}{2}x_2 + 2 - \frac{x_3}{2}$$

$$= \frac{3}{2}x_3 + 2 - \frac{x_3}{2} = 2 - x_3$$

$\boxed{x_1 = 2 - x_3}$

x_3 any value!

INFINITELY MANY SOLUTIONS.