

## Inequalities

### Collaborators:

We are familiar with equalities such as  $x + 2 = 4$  and  $(x + y)^2 = x^2 + 2xy + y^2$ . Quite often in mathematics, we are less certain of the values we are working with. In these cases, we may use **inequalities**. An inequality is a statement about the order of two objects.

In  $\mathbb{R}$ , there is a natural order given by our usual sense of which numbers are bigger than others. For example, the statement  $2 < 3$  is the statement that the number 2 is less than the number 3. We can also write  $2 \leq 2$  to mean that 2 is less than or equal to 2. There are also the statements  $3 > 2$  and  $2 \geq 2$ , which indicate “greater than” instead of “less than”.

Exercise: Consider the inequality  $x + 3 \geq \pi$ . Find all  $x \in \mathbb{R}$  for which this inequality is satisfied. Prove your assertion.

### Scratch Work

*Proof.*

□

Some properties of inequalities of real numbers:

- a) If  $x \in \mathbb{R}$ , then exactly one of the following is true:  $x > 0$  or  $x < 0$  or  $x = 0$ .
- b) If  $x > y$ , then  $-x < -y$ .
- c) If  $x > y$  and  $c \in \mathbb{R}$ , then  $x + c > y + c$ .
- d) If  $x > 0$  and  $y > 0$ , then  $xy > 0$ .
- e) If  $x > y$  and  $y > z$ , then  $x > z$ .

Exercise: Show that if  $x > 0$  and  $y < 0$ , then  $xy < 0$ .

**Scratch Work**

*Proof.*

□

Exercise: Find all  $x \in \mathbb{R}$  such that  $-3x^2 + 4x \geq 1$ .

**Scratch Work**

Writeup:

We define the **modulus** or **absolute value** of a real number  $x$  as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

Exercise: Find all  $x \in \mathbb{R}$  such that  $|x + 5| \geq 1$ .

**Scratch Work**

Writeup:

One incredibly important inequality is the Triangle Inequality, which states that for all real numbers  $x, y \in \mathbb{R}$ , we have

$$|x + y| \leq |x| + |y|.$$

Exercise: Use the Triangle Inequality to prove the Reverse Triangle Inequality: for all  $x, y \in \mathbb{R}$ ,

$$|x - y| \geq \left| |x| - |y| \right|.$$

**Scratch Work**

*Proof.*

□