Math 8
Worksheet
Week 4, Thursday

Induction

Collaborators:

We started talking about induction last time. The **Principle of Mathematical Induction** says that in order to prove a family of statements $P(n)$ for $n \in \mathbb{N}$, it is sufficient to prove $P(1)$ and to prove that $P(n)$ implies $P(n+1)$. Induction is a powerful proof technique that allows us to tackle some problems that might otherwise not know how to reason about. If we're not careful though, it is also possible to get ourselves in trouble.

Theorem. *For all positive integers n, we have $n+1 < n$.*

Proof. We proceed by induction on n. Fix some positive integer n and assume $n+1 < n$. Then, adding 1 to both sides, we have

$$n+2 < n+1,$$

proving that $(n+1)+1 < (n+1)$. By the Principle of Mathematical Induction, we conclude that for all positive integers n, it is true that $n+1 < n$. □

What went wrong?

Theorem. *All real numbers are equal.*

Proof. [a] We proceed by induction on the statement $P(n)$ that

"For any real numbers $a_1, a_2, \ldots, a_n$, we have $a_1 = a_2 = \cdots = a_n$."

Base Case: In the case $n = 1$, we see the statement $P(1)$ is true.

Inductive Step: Let $k \in \mathbb{N}$ be given and assume $P(k)$, the statement that for any list of real numbers $a_1, \ldots, a_k$, we have $a_1 = \cdots = a_k$. We wish to show $P(k+1)$.
Consider any collection of real numbers $a_1, \ldots, a_{k+1}$. Let us now consider the first k elements $a_1, \ldots, a_k$. By our inductive hypothesis, we know

$$a_1 = \cdots = a_k. \tag{1}$$

Now, take the last k elements $a_2, \ldots, a_{k+1}$. By the inductive hypothesis, we know

$$a_2 = \cdots = a_{k+1}. \tag{2}$$

Now, by transitivity of equality of real numbers, (1) and (2) yield

$$a_1 = \cdots = a_{k+1},$$

as desired.

Conclusion: By the Principle of Mathematical Induction, we have that for any list of n real numbers $a_1, \ldots, a_n$, it must be that $a_1 = \cdots a_n$. Since the a_i were chosen arbitrarily, we conclude that all real numbers are equal. □

[a] Credit to A. J. Hildebrand

Wat.

Theorem. *All positive integers are equal.*

Proof. [a] Consider the statement

$$\text{“For any } x, y \in \mathbb{N}, \text{ if } \max(x, y) = n, \text{ then } x = y,\text{”}$$

where $\mathbb{N}$ is the set of all positive integers. We shall prove this statement by induction on n.

Base Case: In the case $n = 1$, we have $x, y \in \mathbb{N}$ such that $\max(x, y) = 1$. The only option in this case is that $x = y = 1$, so we have $x = y$.

Inductive Step: Let $k \in \mathbb{N}$ be given and suppose the statment holds for $n = k$. We wish to show that it holds for $n = k + 1$.
Let $x, y \in \mathbb{N}$ such that $\max(x, y) = k + 1$. Then

$$\begin{aligned} \max(x-1, y-1) &= \max(x, y) - 1 \\ &= (k+1) - 1 \\ &= k. \end{aligned}$$

By the inductive hypothesis, we see that $x - 1 = y - 1$. Adding 1 to both sides gives $x = y$, thereby completing the inductive step.

Conclusion: Using the Principle of Mathematical Induction, we have shown that any two positive integers are equal. Since the integers x, y are chosen arbitrarily, this is true for all $x, y \in \mathbb{N}$. □

[a] Credit to A. J. Hildebrand

Well, I guess all of math is just trivial.

Exercise [Liebeck Ch 8. Problem 1]: You are in charge of designing a currency, the Gallifreyan penny, but the supreme leader of Gallifrey really likes the number 3 and wants there to be 3 gp coins. Show that it is possible to pay, without requiring change, any whole number of gp greater than 7 using only 3 gp and 5 gp coins.
[In fact, this is a special case of a result in Number Theory called Bézout's Lemma, which says that for integers x, y with greatest common divisor d, the set of all numbers that can be written as integer linear combinations $ax + by$ are exactly the multiples of d.]

Scratch Work

Proof.

□

Exercise: Given a prime number p, show that for any $n \in \mathbb{N}$, p divides one of the numbers n, $n+1$, $\dots$, $n+(p-1)$.

Scratch Work

Proof.

$\square$

Occasionally, we encounter statements that are difficult to prove inductively because there is no clear connection between the $n+1$-th case and the n-th case, but there might be a more clear connection between the $n+1$-th case and the k-th case for some $1 \leq k < n$. In such a scenario, we may use the **Principle of Strong Mathematical Induction**. The Principle of Strong Mathematical Induction says that in order to prove a family of statements $P(n)$ for $n \in \mathbb{N}$, it is sufficient to prove $P(1)$ and to prove that $P(1), \ldots, P(n)$ together imply $P(n+1)$.

Exercise: Just for this problem, count 1 as a prime number. A well-known result in number theory says that for every integer $x \geq 3$, there is a prime number p such that $\frac{1}{2}x < p < x$. Using this result and strong induction, prove that every positive integer is equal to a sum of primes, all of which are different.

Scratch Work

Proof.

□