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Math 8
Worksheet
Week 9, Tuesday

Cartesian Produccts, Equivalence Relations

Collaborators:

We now discuss an important way to talk about different elements of a set being equivalent. To do so, we will first need the notion of a Cartesian product. Given two sets A, B, the **Cartesian product** of A and B is the set

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

of all ordered pairs, where the first component is chosen from A and the second component is chosen from B.

Examples. Describe the set $A \times B$ in each of the following cases.

(a) $A = \{1, 2, 3\}$ and $B = \{4, 5\}$.

(b) $A = B = \mathbb{R}$.

(c) $A = \mathbb{Q}$, $B = \mathbb{R}$.

We may now define a way of relating two objects in a set. A **relation** R on a set S is any subset $R \subseteq S \times S$. If $s, t \in S$ and $(s, t) \in R$, then we say s is related to t and write sRt or $s \sim t$. When context makes it clear what we are talking about, the relation R is sometimes just called the relation $\sim$.

Examples. For each of the following, say explicitly what R is given the description of the relation.

(a) $S = \{1, 2, 3, 4, 5\}$ with $1 \sim 1$, $1 \sim 2$, $1 \sim 3$, $4 \sim 5$, and $5 \sim 4$

(b) $S = \mathbb{R}$ with $s \sim t$ if and only if $s = t$.

(c) $S = \mathbb{R}$ with $s \nsim t$ for all $s, t \in \mathbb{R}$.

(d) $S = \mathbb{R}$ with $s \sim t$ for all $s, t \in \mathbb{R}$.

Finally, we may define a notion of equivalence. We say that a relation R on the set S is an **equivalence relation** if R satisfies the following three properties.

(i) Reflexive: $s \sim s$ for all $s \in S$. Alternatively, we could say $(s, s) \in R$ for all $s \in S$.

(ii) Symmetric: For all $s, t \in S$, if $s \sim t$, then $t \sim s$. Alternatively, we could say $(s, t) \in R$ implies $(t, s) \in R$.

(iii) Transitive: For all $s, t, u \in S$, if $s \sim t$ and $t \sim u$, then $s \sim u$. Alternatively, we could say $(s, t), (t, u) \in R$ implies $(s, u) \in R$.

One might think of these as the three properties one would want to have if one is to say that some set of things are equivalent. It would be good if every object was equivalent to itself. It would be good if whenever a is equivalent to b, then b is also equivalent to a. If a is equivalent to b and b is equivalent to c, it would be good if a were equivalent to c.

Examples. Say which of these three properties are satisfied by the relations defined on the previous page. Which relations are equivalence relations?

Now that we have defined a notion of equivalence between different elements of a set, we can group the elements according to which ones are equivalent. Given an equivalence relation $\sim$ on a set S, the **equivalence class** of an element $s \in S$ is defined to be

$$\mathrm{cl}(s) = \{t \in S \mid t \sim s\}$$

Other common notations for $\mathrm{cl}(s)$ are $[s]$ and $\overline{s}$. Note that because $\sim$ is an equivalence relation, we always have $s \in \mathrm{cl}(s)$ and $\mathrm{cl}(t) = \mathrm{cl}(s)$ if $t \in \mathrm{cl}(s)$. More generally, we can see that the set of equivalence classes **partions** the set S. That is, separate equivalence classes share no elements and the union of all equivalence classes is S. It is useful to think of the set of equivalence classes as a new set where all the old elements deemed to be equivalent have been smooshed down to one object: their eequivalence class.

Example. Write the set of equivalence classes for each of the equivalence relations in the previous problem.

Important Example. Show that the relation $x \sim y$ if and only if $x \equiv y \pmod 5$ is an equivalence relation on $\mathbb{Z}$. What are the equivalence classes?