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Math 8
Worksheet
Week 9, Tuesday

Cartesian Producets, Equivalence Relations

Collaborators:

We now discuss an important way to talk about different elements of a set being equivalent. To do so, we will first need the notion of a Cartesian product. Given two sets A, B, the **Cartesian product** of A and B is the set

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

of all ordered pairs, where the first component is chosen from A and the second component is chosen from B.

Examples. Describe the set $A \times B$ in each of the following cases.

(a) $A = \{1, 2, 3\}$ and $B = \{4, 5\}$.

(b) $A = B = \mathbb{R}$.

(c) $A = \mathbb{Q}$, $B = \mathbb{R}$.

(a) $A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$.

(b) $A \times B = \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$. This is often thought of as the coordinate plane.

(c) $A \times B = \{(x, y) \mid x \in \mathbb{Q}, y \in \mathbb{R}\}$. This is the subset of $\mathbb{R}^2$ composed of all points with rational first coordinate. Equivalently, it is the collection of all vertical lines at rational points on the x-axis.

We may now define a way of relating two objects in a set. A **relation** R on a set S is any subset $R \subseteq S \times S$. If $s, t \in S$ and $(s,t) \in R$, then we say s is related to t and write sRt or $s \sim t$. When context makes it clear what we are talking about, the relation R is sometimes just called the relation $\sim$.

Examples. For each of the following, say explicitly what R is given the description of the relation.

(a) $S = \{1, 2, 3, 4, 5\}$ with $1 \sim 1$, $1 \sim 2$, $1 \sim 3$, $4 \sim 5$, and $5 \sim 4$

(b) $S = \mathbb{R}$ with $s \sim t$ if and only if $s = t$.

(c) $S = \mathbb{R}$ with $s \nsim t$ for all $s, t \in \mathbb{R}$.

(d) $S = \mathbb{R}$ with $s \sim t$ for all $s, t \in \mathbb{R}$.

(a) R is the subset $\{(1,1), (1,2), (1,3), (4,5), (5,4)\} \subset S \times S$.

(b) R is the diagonal subset $\{(x,x) \mid x \in \mathbb{R}\} \subset S \times S$.

(c) Since no pairs are related, R is the empty set, $\emptyset \subset S \times S$.

(d) Since every pair is related, we have $R = S \times S$.

Finally, we may define a notion of equivalence. We say that a relation R on the set S is an **equivalence relation** if R satisfies the following three properties.

(i) Reflexive: $s \sim s$ for all $s \in S$. Alternatively, we could say $(s, s) \in R$ for all $s \in S$.

(ii) Symmetric: For all $s, t \in S$, if $s \sim t$, then $t \sim s$. Alternatively, we could say $(s, t) \in R$ implies $(t, s) \in R$.

(iii) Transitive: For all $s, t, u \in S$, if $s \sim t$ and $t \sim u$, then $s \sim u$. Alternatively, we could say $(s, t), (t, u) \in R$ implies $(s, u) \in R$.

One might think of these as the three properties one would want to have if one is to say that some set of things are equivalent. It would be good if every object was equivalent to itself. It would be good if whenever a is equivalent to b, then b is also equivalent to a. If a is equivalent to b and b is equivalent to c, it would be good if a were equivalent to c.

Examples. Say which of these three properties are satisfied by the relations defined on the previous page. Which relations are equivalence relations?

(a) This relation does not satisfy reflexivity because $2 \nsim 2$. This relation fails to satisfy symmetry since $1 \sim 2$, but $2 \nsim 1$. Finally, R fails to be transitive because $4 \sim 5$ and $5 \sim 4$, but $4 \nsim 4$.

(b) This relation satisfies all three, and is an equivalence relation!

(c) This relation fails reflexivity, but satisfies both symmetry and transitivity.

(d) This relation also satisfies all three and is an equivalence relation, but it's not very interesting. It is just saying to let every point be equivalent.

Now that we have defined a notion of equivalence between different elements of a set, we can group the elements according to which ones are equivalent. Given an equivalence relation $\sim$ on a set S, the **equivalence class** of an element $s \in S$ is defined to be

$$\mathrm{cl}(s) = \{t \in S \mid t \sim s\}$$

Other common notations for $\mathrm{cl}(s)$ are $[s]$ and $\overline{s}$. Note that because $\sim$ is an equivalence relation, we always have $s \in \mathrm{cl}(s)$ and $\mathrm{cl}(t) = \mathrm{cl}(s)$ if $t \in \mathrm{cl}(s)$. More generally, we can see that the set of equivalence classes **partions** the set S. That is, separate equivalence classes share no elements and the union of all equivalence classes is S. It is useful to think of the set of equivalence classes as a new set where all the old elements deemed to be equivalent have been smooshed down to one object: their eequivalence class.

Example. Write the set of equivalence classes for each of the equivalence relations in the previous problem.

(b) Since each element is equivalent only to itself, we have $\mathrm{cl}(x) = \{x\}$ for all $x \in \mathbb{R}$. Thus, the set of equivalence classes is

$$\Big\{ \mathrm{cl}(x) \mid x \in \mathbb{R} \Big\} = \Big\{ \{x\} \mid x \in \mathbb{R} \Big\}.$$

Note how this set is a set of sets, but is otherwise not meaningfully different from $\mathbb{R} = \{x \mid x \in \mathbb{R}\}$ itself. Since our equivalence relation did not introduce any equivalences we did not already have in $\mathbb{R}$, the set of equivalence classes is no smaller than the original set we started with.

(d) Since all elements of $\mathbb{R}$ are equivalent under this equivalence relation, we may just pick one, say 0, and write $\mathrm{cl}(0) = \mathbb{R}$. The set of equivalence classes is then

$$\Big\{ \mathrm{cl}(0) \Big\} = \Big\{ \mathbb{R} \Big\}.$$

In this case, we have trivialised the original set by sending all of its elements to a single equivalence class!

Important Example. Show that the relation $x \sim y$ if and only if $x \equiv y \pmod 5$ is an equivalence relation on $\mathbb{Z}$. What are the equivalence classes?

I leave the proof to you.

We already know that, modulo 5, every integer is congruent to 0, 1, 2, 3, or 4. It is unsurprising, then, that the equivalence classes are

$$\begin{aligned}
\overline{0} &= \{n \in \mathbb{Z} \mid 5 \mid n\}, \\
\overline{1} &= \{n \in \mathbb{Z} \mid 5 \mid n-1\}, \\
\overline{2} &= \{n \in \mathbb{Z} \mid 5 \mid n-2\}, \\
\overline{3} &= \{n \in \mathbb{Z} \mid 5 \mid n-3\}, \\
\overline{4} &= \{n \in \mathbb{Z} \mid 5 \mid n-4\}.
\end{aligned}$$

These equivalence classes are familiar. They are exactly the elements of $\mathbb{Z}_5$. That is, $\mathbb{Z}_5$ is just the set of equivalence classes of $\mathbb{Z}$ under the relation $x \sim y$ if and only if $x \equiv y \pmod 5$.