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Math 8
Worksheet
Week 9, Tuesday

Functions

Collaborators:

As you transition to higher mathematics, you will see that math is often the study of functions. It is perhaps time, then, to define the word. Let S and T be sets. A **function** (or a **map**) $f\colon S \to T$ is a rule that assigns to each element $s \in S$ exactly one element $f(s) \in T$.

The element $f(s) \in T$ is called the **image** of s under f. It is common say "f sends s to $f(s)$" or "f maps s to $f(s)$" and to write $s \mapsto f(s)$.

The set S is called the **domain** of f, and the set T is called the **codomain** of f. The set $f(S) = \{f(s) \mid s \in S\}$, the set of all possible values of $f(s)$ with $s \in S$, is called the **image** (or the **range**) of f.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Identify the domain, codomain, and image of f.

The domain and codomain of f are both $\mathbb{R}$ since f is a function $\mathbb{R} \to \mathbb{R}$. The image of f is $f(\mathbb{R}) = \{x^2 \mid x \in \mathbb{R}\}$, the set of all nonnegative real numbers.

Notice that, in general, the range of f is a subset of the codomain of f. For some functions, the range will actually be the entire codomain. A function $f\colon S \to T$ is called **onto** (or **surjective**) if $f(S) = T$. That is, f is onto if for all $t \in T$, there exists $s \in S$ such that $f(s) = t$.
A function $f\colon S \to T$ is called **one-to-one** (or 1-1 or **injective**) if no two elements of S are mapped to the same element of T. That is, f is one-to-one if $f(s_1) = f(s_2)$ implies $s_1 = s_2$.
We say that f is **bijective** (or a bijection) if f is both onto and one-to-one.

Examples.

(a) Consider the function $f\colon \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(m, n, r) = 2^m 3^n 6^r$. Is f onto? Is f 1-1?

(b) Find an onto function from $\mathbb{N}$ to $\mathbb{Z}$.

(a) f is not onto because no choice of m, n, r will give $f(m, n, r) = 2^m 3^n 6^r = 7$, but $7 \in \mathbb{N}$. f is also not 1-1 because $f(1, 1, 2) = f(2, 2, 1)$ even though $(1, 1, 2) \neq (2, 2, 1)$.

(b) Consider the function which maps

$$\begin{aligned} 1 &\mapsto 0, \\ 2 &\mapsto 1, \\ 3 &\mapsto -1, \\ 4 &\mapsto 2, \\ 5 &\mapsto -2, \\ &\vdots \end{aligned}$$

Given a function $f\colon S \to T$ and an element $s \in S$, we may map s to $f(s)$. Since $f(s)$ is simply some element of T, it is perhaps natural to consider mapping it again to some other set. If we have a function $g\colon T \to U$, we can apply f and then g to map $s \mapsto f(s) \mapsto g(f(s))$. The **composition** of f and g is the function $g \circ f\colon S \to U$ defined by

$$(g \circ f)(s) = g(f(s)).$$

Exercise. Let S, T, U be sets and let $f\colon S \to T$, $g\colon T \to U$ be functions. Though they can be found in the textbook, it is a good exercise to prove the following facts by yourself.

(i) If f and g are both one-to-one, then so is $g \circ f$.

(ii) If f and g are both onto, then so is $g \circ f$.

(iii) If f and g are both bijective, then so is $g \circ f$.

The proofs can be found in the textbook.