IRREDUCIBLE COMPONENTS OF VARIETIES OF REPRESENTATIONS. THE LOCAL CASE

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Abstract. For any positive integer d, we determine the irreducible components of the varieties that parametrize the d-dimensional representations of a local truncated path algebra Λ. Here Λ is a quotient $KQ/\langle$the paths of length $L+1\rangle$ of a path algebra KQ, where K is an algebraically closed field, L is a positive integer, and Q is the quiver with a single vertex and a finite number r of loops. The components are determined in both the classical and the Grassmannian settings, $\mathbf{Rep}_d(\Lambda)$ and $\mathrm{GRASS}_d(\Lambda)$. Our method is to corner the components by way of a twin pair of upper semicontinuous maps from $\mathbf{Rep}_d(\Lambda)$ to a poset consisting of sequences of semisimple modules.

An excerpt of the main result is as follows. Given a sequence $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ of semisimple modules with $\dim \bigoplus_{0 \le l \le L} \mathbb{S}_l = d$, let $\mathbf{Rep}\,\mathbb{S}$ be the subvariety of $\mathbf{Rep}_d(\Lambda)$ consisting of the points that parametrize the modules with radical layering $\mathbb{S}$. (The radical layering of a Λ-module M is the sequence $\left(J^l M / J^{l+1} M\right)_{0 \le l \le L}$, where J is the Jacobson radical of Λ.) Suppose the quiver Q has $r \ge 2$ loops. If $d \le L+1$, the variety $\mathbf{Rep}_d(\Lambda)$ is irreducible and, generically, its modules are uniserial. If, on the other hand, $d > L+1$, then the irreducible components of $\mathbf{Rep}_d(\Lambda)$ are the closures of the subvarieties $\mathbf{Rep}\,\mathbb{S}$ for those sequences $\mathbb{S}$ which satisfy the inequalities $\dim \mathbb{S}_l \le r \cdot \dim \mathbb{S}_{l+1}$ and $\dim \mathbb{S}_{l+1} \le r \cdot \dim \mathbb{S}_l$ for $0 \le l < L$; generically, the modules in any such component have socle layering $(\mathbb{S}_L, \dots, \mathbb{S}_0)$. As a byproduct, the main result provides further installments of generic information on the modules corresponding to the irreducible components of the parametrizing varieties.

1. Introduction and main result

The overarching objective is to generically organize the representation theory of basic finite dimensional algebras Λ over an algebraically closed field K. This amounts to tackling the following two sequential tasks for each dimension vector $\mathbf{d}$ of Λ: (**A**) Determine, in both geometric and representation-theoretic terms, the irreducible components of the affine parametrizing varieties $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$ for the $\mathbf{d}$-dimensional Λ-modules, and (**B**) explore generic features of the modules corresponding to the individual components, such as generic decomposition properties, generic tops, socles, submodule lattices, etc.

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The generic approach to representation theory was pioneered by Kac and further developed by Schofield in the hereditary case, that is, when $\Lambda = KQ$ is a path algebra; see [**16**, **17**] and [**21**]. In this situation, the parametrizing varieties of the modules with any fixed dimension vector $\mathbf{d}$ are affine spaces (hence irreducible), and the task is reduced to (**B**), i.e., to relating generic features of the $\mathbf{d}$-dimensional modules to the quiver Q.

On departing from the hereditary scenario, one is confronted with the initial roadblock (major) that, typically, the parametrizing varieties $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$ are reducible and sport a highly complex geometry. So pinning down the components in a representation-theoretically useful format inevitably becomes the first step in any non-hereditary generic representation theory. Hence, the goal of determining these components from Q and I was singled out early on, by Kraft in [**18**] at the latest. Yet, so far, full solutions to the component problem outside the hereditary setting have been limited to classes of tame algebras, for which the structure of the finite dimensional representations is fully understood. We will follow up with a more complete overview of prior work at the end of the introduction.

Our main result, stated below, addresses *truncated path algebras*, that is, algebras of the form $\Lambda = KQ/\langle \text{all paths of length } L+1\rangle$ for a quiver Q and a fixed positive integer L. This class includes the basic hereditary finite dimensional algebras and those with vanishing radical square. To indicate the prominent place held by truncated path algebras, we note that every basic finite dimensional algebra Λ' is a factor algebra of a unique truncated path algebra Λ which has the same quiver and Loewy length as Λ'; in this pairing, each $\mathrm{Rep}_{\mathbf{d}}(\Lambda')$ is a subvariety of $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$, which makes the truncated case a natural first target in the quest for generic information.

In case Λ is truncated, one always has a supply of comparatively large irreducible subvarieties of $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$ at one's disposal (see [**1**, Section 5]), namely, the varieties labeled $\mathbf{Rep}\,\mathbb{S}$, to be introduced next. The *radical layering* of a Λ-module M is the sequence $\mathbb{S}(M) = \big(J^l M/J^{l+1}M\big)_{0\le l\le L}$, where J is the Jacobson radical of Λ (satisfying $J^{L+1} = 0$ in our setup); the *socle layering* $\mathbb{S}^*(M)$ is defined dually. Both layerings are *semisimple sequences with dimension vector* $\underline{\dim}\,M$, i.e., sequences of the form $\mathbb{S} = (\mathbb{S}_0, \mathbb{S}_1, \dots, \mathbb{S}_L)$ whose entries $\mathbb{S}_l$ are semisimple modules such that $\underline{\dim}\,\mathbb{S} := \sum_{0\le l\le L} \underline{\dim}\,\mathbb{S}_l$ equals $\underline{\dim}\,M$. Identifying isomorphic semisimple modules, we define $\mathbf{Rep}\,\mathbb{S}$ to be the locally closed subvariety of $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$ that consists of the points corresponding to modules with radical layering $\mathbb{S}$; thus $\mathbf{Rep}\,\mathbb{S}$ is pinned down by an $(L+1)\times|Q_0|$-matrix of nonnegative integers.

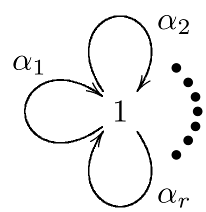
The subvarieties $\mathbf{Rep}\,\mathbb{S}$ partition $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$ into finitely many strata. Since any irreducible component of $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$ intersects precisely one of them in a nontrivial open set, all of the components are among the closures $\overline{\mathbf{Rep}\,\mathbb{S}}$. Task (**A**) above therefore amounts to characterizing those semisimple sequences $\mathbb{S}$ which arise as the generic radical layerings of the irreducible components of $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$. For the purpose of filtering out the relevant sequences $\mathbb{S}$, we consider the "dominance" partial order on the set of semisimple sequences (see under Conventions below), and use the following observation as a starting point.

Observation. (cf. Theorem 3.1) *Let Λ be truncated. If $\mathbb{S} = \mathbb{S}(M)$ for some module M with the property that the pair $\big(\mathbb{S}(M), \mathbb{S}^*(M)\big)$ is a minimal element of the set*

$$\mathbf{rad\text{-}soc}(\mathbf{d}) := \{\big(\mathbb{S}(N), \mathbb{S}^*(N)\big) \mid N \in \Lambda\text{-mod},\ \underline{\dim}\,N = \mathbf{d}\},$$

then the closure $\overline{\mathbf{Rep}\,\mathbb{S}}$ is an irreducible component of $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$.

The converse fails for general truncated Λ, as witnessed by Example 4.8. But in a number of significant special cases it does hold, notably when Λ is local, meaning that the quiver Q has the form



for some $r \in \mathbb{N}$. Since Q has only one vertex in the local situation, the dimension vector $\mathbf{d}$ of a module is replaced by its dimension d. In this case, $\mathbf{Rep}_d(\Lambda)$ consists of the r-tuples $(A_1, \dots, A_r)$ of $d \times d$-matrices subject to the condition that all $(L+1)$-fold products of the A_i vanish, while $\mathrm{GRASS}_d(\Lambda)$ is the closed subvariety of the classical Grassmann variety $\mathrm{Gr}\big((\dim \Lambda^d) - d,\, \Lambda^d\big)$ which picks out the Λ-sub*modules* of codimension d of the free left Λ-module Λ^d.

In case $r = 1$, Λ is a truncated polynomial ring in a single variable, and all of the varieties $\mathbf{Rep}_d(\Lambda)$ are trivially irreducible. Otherwise, we find:

Main Theorem. *Let Λ be a local truncated path algebra of Loewy length $L+1$, whose quiver Q has $r \geq 2$ loops, and let d be a positive integer.*

(I) *If $d > L+1$ and $\mathbb{S}$ is a semisimple sequence of dimension d, the following conditions are equivalent:*

(1) *The closure $\overline{\mathbf{Rep}\,\mathbb{S}}$ is an irreducible component of $\mathbf{Rep}_d(\Lambda)$.*

(1′) *The closure $\overline{\mathrm{GRASS}\,\mathbb{S}}$ is an irreducible component of $\mathrm{GRASS}_d(\Lambda)$.*

(2) $\dim \mathbb{S}_l \leq r \cdot \dim \mathbb{S}_{l-1}$ *and* $\dim \mathbb{S}_{l-1} \leq r \cdot \dim \mathbb{S}_l$ *for $l \in \{1, \dots, L\}$; in particular, $\mathbb{S}_l \neq 0$ for all $l \leq L$.*

(3) $\mathbf{Rep}\,\mathbb{S} \neq \varnothing$, *and $\mathbb{S}^* = (\mathbb{S}_L, \mathbb{S}_{L-1}, \dots, \mathbb{S}_0)$ is the generic socle layering of the modules in $\mathbf{Rep}\,\mathbb{S}$.*

(4) $\mathbb{S} = \mathbb{S}(M)$ *for some minimal pair $\big(\mathbb{S}(M), \mathbb{S}^*(M)\big)$ in* $\mathbf{rad\text{-}soc}(\mathbf{d})$.

Moreover, the irreducible components of $\mathbf{Rep}_d(\Lambda)$ are precisely the $\overline{\mathbf{Rep}\,\mathbb{S}}$ for semisimple sequences $\mathbb{S}$ satisfying the above conditions.

(II) *If, on the other hand, $d \leq L+1$, the variety $\mathbf{Rep}_d(\Lambda)$ is irreducible and, generically, its modules are uniserial. In this situation, conditions* (1), (1′) *and* (4) *are equivalent.*

For $L = 1$ and $r = 2$, the irreducible components of the varieties $\mathbf{Rep}_d(\Lambda)$ were already determined by Donald and Flanigan [**7**], as well as by Morrison [**19**]. For arbitrary choices of r, the case $L = 1$ is covered by [**3**, Theorem 3.12].

Condition (2) of the theorem permits us to list – without any computational effort – the irreducible components of $\mathbf{Rep}_d(\Lambda)$, tagged by their generic radical layerings. Moreover, the theorem provides further generic mileage, i.e., progress towards Task (**B**) above, beyond the installment stated in condition (3); see Section 4.

In [**15**], it is shown that the equivalence "(1) $\iff$ (4)" remains valid for truncated path algebras based on an acyclic quiver Q; however, the remaining equivalences of the theorem fail outside the local case in general, even under the assumption of acyclicity of Q. As mentioned above, Example 4.8 shows the implication "(1) $\Longrightarrow$ (4)" to fail over general truncated path algebras.

Section 2 applies to arbitrary basic finite dimensional algebras. It provides the framework for our technique of cornering the irreducible components of $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$ and $\mathrm{GRASS}_{\mathbf{d}}(\Lambda)$ by way of suitable upper semicontinuous maps. Section 3 narrows the focus to truncated path algebras, preparing for both applications and a proof of the Main Theorem. In Section 4, we derive consequences from this theorem and buttress the theory with examples. A proof of the Main Theorem is given in Section 5.

Finally, we place the above theorem into the context of existing work in non-hereditary scenarios. Results of Crawley-Boevey and Schröer in [**6**] target those irreducible components of $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$ (for arbitrary finitely generated Λ) whose representations are generically decomposable, relating them to components for smaller dimension vectors. Moreover, Babson, Thomas and the author showed how to obtain (for all choices of Q and I) a finite list of representation-theoretically defined closed irreducible subvarieties of $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$ which includes the irreducible components (see [**1**, Sections 3-5]). This turns the component problem into a sifting problem even beyond the truncated case, but the sifting then becomes more involved in that the varieties $\mathbf{Rep}\,\mathbb{S}$ in turn break up into multiple irreducible components in general. For vanishing radical square, task (**A**) was recently completed by Bleher, Chinburg and the author, and task (**B**) was pushed to the level achieved for hereditary algebras (see [**3**]). Beyond that, a full description of the irreducible components is available for certain special biserial algebras, notably for the family of Gelfand-Ponomarev algebras $K[X,Y]/(XY, X^r, Y^s)$ (Schroër [**22**]), next to the algebra $K[X,Y]/(X^2, Y^2)$ (Riedtmann-Rutscho-Smalø [**20**]) and gentle special biserial algebras (Carroll-Weyman [**5**]). These algebras have tame representation type, and explicit descriptions of their finitely generated indecomposable modules help access the components of their parametrizing varieties. Further types of tame algebras were addressed by Barot-Schröer in [**2**] and Geiss-Schröer in [**9**, **10**]. We point out that interest in varieties which consist of sequences of matrices satisfying certain relations is not limited to the role they play in the representation theory of algebras; see, e.g., [**11**], [**8**], [**12**].

Conventions: For our technique of graphing Λ-modules, we refer to [**1**, Definition 3.9 and subsequent examples]. Throughout, Λ is a basic finite dimensional algebra over an algebraically closed field K, and Λ-mod (resp. mod-Λ) the category of finitely generated left (resp. right) Λ-modules. By J we denote the Jacobson radical of Λ; say $J^{L+1} = 0$. Without loss of generality, we assume that $\Lambda = KQ/I$ for some quiver Q and admissible ideal $I \subseteq KQ$. By our convention, the product pq of two paths p, q in KQ is the concatenation "p after q" if the starting vertex of p equals the terminal vertex of q, and is 0 otherwise.

The set $Q_0 = \{e_1, \dots, e_n\}$ of vertices of Q will be identified with a full set of primitive idempotents of Λ. Hence, the simple left Λ-modules are $S_i = \Lambda e_i / J e_i$, $1 \le i \le n$, up to isomorphism. Unless we want to distinguish among different embeddings, we systematically identify isomorphic semisimple modules; in other words, we identify finitely generated semisimples with their dimension vectors. This provides us with a partial order on the set of finite dimensional semisimple modules: Namely, $U \subseteq V$ if and only if $\underline{\dim}\, U \le \underline{\dim}\, V$ under the componentwise partial order on $\mathbb{N}_0^n$.

Let $\mathbb{S}$ be a semisimple sequence, that is, a sequence of the form $\mathbb{S} = (\mathbb{S}_0, \mathbb{S}_1, \dots, \mathbb{S}_L)$ such that each $\mathbb{S}_l$ is a semisimple module, and define $\underline{\dim}\, \mathbb{S} = \sum_{0 \le l \le L} \underline{\dim}\, \mathbb{S}_l$. When $\mathbb{S}_l = 0$ for all $l \ge m+1$, we will also write $\mathbb{S}$ in the clipped form $(\mathbb{S}_0, \dots, \mathbb{S}_m)$. The set $\mathfrak{Seq}(\mathbf{d})$ of all semisimple sequences with fixed dimension vector $\mathbf{d}$ is endowed with the following partial order, dubbed the *dominance order*:

$$\mathbb{S} \le \mathbb{S}' \quad \iff \quad \bigoplus_{0 \le j \le l} \mathbb{S}_j \subseteq \bigoplus_{0 \le j \le l} \mathbb{S}'_j \ \text{ for } \ l \in \{0, 1, \dots, L\}.$$

An element x of a Λ-module M is said to be *normed* if $x = e_i x$ for some i. A *top element* of M is a normed element in $M \setminus JM$, and a *full sequence of top elements of* M is any generating set of M consisting of top elements which are K-linearly independent modulo JM.

Given a subset $\mathfrak{U}$ of $\mathbf{Rep}_\mathbf{d}(\Lambda)$, the modules corresponding to the points in $\mathfrak{U}$ are called the modules "in" $\mathfrak{U}$. When $\mathfrak{U}$ is irreducible, the modules in $\mathfrak{U}$ are said to *generically have property* $(*)$ in case all modules in some dense open subset of $\mathfrak{U}$ satisfy $(*)$. As we will see below, radical layerings and socle layerings of modules are generically constant on any irreducible subvariety of $\mathbf{Rep}_\mathbf{d}(\Lambda)$. Hence it is meaningful to speak of *the* generic radical and socle layerings of the irreducible components of $\mathbf{Rep}_\mathbf{d}(\Lambda)$.

2. Facts for arbitrary basic algebras Λ

2.A. The parametrizing varieties.

We recall the definitions of the relevant varieties parametrizing the isomorphism classes of (left) Λ-modules with dimension vector $\mathbf{d} = (d_1, \dots, d_n)$.

The affine setting: The classical affine variety is $\mathbf{Rep}_\mathbf{d}(\Lambda) =$

$$\Big\{(x_\alpha)_{\alpha \in Q_1} \in \prod_{\alpha \in Q_1} \operatorname{Hom}_K\big(K^{d_{\text{start}(\alpha)}},\, K^{d_{\text{end}(\alpha)}}\big) \mid \text{the } x_\alpha \text{ satisfy all relations in } I\Big\},$$

where Q_1 is the set of arrows of the quiver Q. This variety carries a conjugation action by $\mathrm{GL}(\mathbf{d}) := \mathrm{GL}_{d_1}(K) \times \cdots \times \mathrm{GL}_{d_n}(K)$, the orbits of which are in bijective correspondence with the isomorphism classes of modules having dimension vector $\mathbf{d}$. Throughout, we denote by $M_x \in \Lambda$-mod the module that corresponds to a point $x \in \mathbf{Rep}_\mathbf{d}(\Lambda)$. If $\mathbb{S}$ is a $\mathbf{d}$-dimensional semisimple sequence, $\mathbf{Rep}\, \mathbb{S}$ stands for the locally closed subvariety consisting of those points $x \in \mathbf{Rep}_\mathbf{d}(\Lambda)$ for which M_x has radical layering $\mathbb{S}$.

The projective setting: Let $d = |\mathbf{d}| = \sum_i d_i$. We fix a projective module $\mathbf{P} = \bigoplus_{1 \le r \le d} \Lambda \mathbf{z}_r$ whose top has dimension vector $\underline{\dim}\,(\mathbf{P}/J\mathbf{P}) = \mathbf{d}$; here $\mathbf{z}_1, \dots, \mathbf{z}_d$ is a full sequence of top elements of $\mathbf{P}$. In other words, $\mathbf{P}$ is a projective cover of $\bigoplus_{1 \le i \le n} S_i^{d_i}$, and thus is the smallest projective Λ-module with the property that every module with dimension vector $\mathbf{d}$ is isomorphic to a quotient $\mathbf{P}/C$ for some submodule $C \subseteq \mathbf{P}$. The variety $\mathrm{Gr}\big((\dim \mathbf{P} - d), \mathbf{P}\big)$ is the vector space Grassmannian of all $(\dim \mathbf{P} - d)$-dimensional K-subspaces of $\mathbf{P}$, and $\mathrm{GRASS}_{\mathbf{d}}(\Lambda)$ is the closed subset consisting of the Λ-submodules C of $\mathbf{P}$ with the property that $\underline{\dim}\, \mathbf{P}/C = \mathbf{d}$. Under the canonical action of the automorphism group $\mathrm{Aut}_\Lambda(\mathbf{P})$ on $\mathrm{GRASS}_{\mathbf{d}}(\Lambda)$, the orbits are again in one-to-one correspondence with the isomorphism classes of modules with dimension vector $\mathbf{d}$. In parallel with the affine setting: For any semisimple sequence $\mathbb{S}$ with dimension vector $\mathbf{d}$, we denote by $\mathrm{GRASS}(\mathbb{S})$ the locally closed subvariety of $\mathrm{GRASS}_{\mathbf{d}}(\Lambda)$ picking out the points C with $\mathbb{S}(\mathbf{P}/C) = \mathbb{S}$. (Caveat: The variety $\mathrm{GRASS}(\mathbb{S})$ introduced here is not to be confused with the much smaller one, $\mathfrak{Grass}\,\mathbb{S}$, used in [**1**], for instance; it is in this smaller variety that information on $\mathrm{GRASS}(\mathbb{S})$ is preferably gleaned.)

Connection between the two settings: The horizontal two-way arrows in the diagram below point to the transfer of geometric information spelled out in the upcoming proposition. It was proved in [**4**], modulo the unirationality statement which was added in [**14**].

$$\begin{array}{ccc} \boxed{\mathrm{GRASS}_{\mathbf{d}}(\Lambda)} & \longleftrightarrow & \boxed{\mathbf{Rep}_{\mathbf{d}}(\Lambda)} \\ \text{(projective)} & & \text{(affine)} \\ \cup\!| & & \cup\!| \\ \boxed{\mathrm{GRASS}(\mathbb{S})} & \longleftrightarrow & \boxed{\mathbf{Rep}\,\mathbb{S}} \\ \text{(quasi-projective)} & & \text{(quasi-affine)} \end{array}$$

Diagram 1.1

Proposition 2.1. Information transfer between the affine and projective settings. *Consider the one-to-one correspondence between the orbits of* $\mathrm{GRASS}_{\mathbf{d}}(\Lambda)$ *and* $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$ *assigning to any orbit* $\mathrm{Aut}_\Lambda(\mathbf{P}).C \subseteq \mathrm{GRASS}_{\mathbf{d}}(\Lambda)$ *the orbit* $\mathrm{GL}(\mathbf{d}).x \subseteq \mathbf{Rep}_{\mathbf{d}}(\Lambda)$ *that represents the same isomorphism class of* Λ*-modules. This correspondence extends to an inclusion-preserving bijection*

$$\Phi : \{\mathrm{Aut}_\Lambda(\mathbf{P})\text{-}\textit{stable subsets of } \mathrm{GRASS}_{\mathbf{d}}(\Lambda)\} \to \{\mathrm{GL}(\mathbf{d})\text{-}\textit{stable subsets of } \mathbf{Rep}_{\mathbf{d}}(\Lambda)\}$$

which preserves and reflects openness, closures, irreducibility, smoothness and unirationality.

In particular, a semisimple sequence $\mathbb{S}$ *is the generic radical layering of the modules parametrized by an irreducible component of* $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$ *precisely when* $\mathbb{S}$ *has the same property relative to an irreducible component of* $\mathrm{GRASS}_{\mathbf{d}}(\Lambda)$.

2.B. Radical and socle layerings.

Recall that the *radical layering* $\mathbb{S}(M)$ of $M \in \Lambda$-mod is the sequence $\big(\mathbb{S}_0(M), \dots, \mathbb{S}_L(M)\big)$ with $\mathbb{S}_l(M) = J^l M / J^{l+1} M$; according to the above conventions, it will be identified with the clipped sequence $\big(\mathbb{S}_0(M), \dots, \mathbb{S}_m(M)\big)$ whenever $J^{m+1}M = 0$ for some $m < L$. Dually, the *socle layering* of M is the sequence $\mathbb{S}^*(M) = \big(\mathbb{S}^*_0(M), \dots, \mathbb{S}^*_L(M)\big)$ with $\mathbb{S}^*_l(M) =$

$\operatorname{soc}_l M/\operatorname{soc}_{l-1} M$. Here $\operatorname{soc}_{-1} M = 0$, and $\operatorname{soc}_0 M \subseteq \operatorname{soc}_1 M \subseteq \cdots \subseteq \operatorname{soc}_L M$ is the standard socle series of M, i.e., $\operatorname{soc}_0 M = \operatorname{soc} M$, and

$$\operatorname{soc}_{l+1} M/\operatorname{soc}_l(M) = \operatorname{soc}(M/\operatorname{soc}_l M).$$

Clearly, $\underline{\dim}\, \mathbb{S}(M) = \underline{\dim}\, \mathbb{S}^*(M) = \underline{\dim}\, M$.

Our interest in semisimple sequences is restricted to those which arise as radical or socle layerings of Λ-modules. Due to the duality addressed in Lemma 2.3 below, we have the choice of prioritizing either radical or socle filtrations. We will usually place the primary focus on radical filtrations, whence the bias in the following definition.

Definition 2.2. A semisimple sequence $\mathbb{S}$ is said to be *realizable* if there exists a left Λ-module M with $\mathbb{S}(M) = \mathbb{S}$.

There is an algorithm for deciding realizability over general path algebras modulo relations. However, in the truncated case, this condition may be checked by mere inspection of the quiver; see Section 3.A. We will freely use the elementary properties of radical and socle layerings assembled in the following lemma.

Lemma 2.3. *Let $M, N \in \Lambda$-mod with $\underline{\dim}\, M = \underline{\dim}\, N$.*

(1) **Duality:** *The radical and socle layerings are dual to each other, in the sense that*

$$\mathbb{S}(D(M)) = \big(D(\mathbb{S}^*_0(M)), \cdots, D(\mathbb{S}^*_L(M)\big) \text{ and } \mathbb{S}^*(D(M)) = \big(D(\mathbb{S}_0(M)), \cdots, D(\mathbb{S}_L(M)\big),$$

where D denotes the duality $\operatorname{Hom}_K(-, K) : \Lambda\text{-mod} \to \text{mod-}\Lambda$.

(2) **Radical layering:** $\underline{\dim}\, J^l M = \underline{\dim} \bigoplus_{l \le j \le L} \mathbb{S}_j(M)$*; in particular*

$$\mathbb{S}(M) \le \mathbb{S}(N) \quad \iff \quad \underline{\dim}\, J^l M \ge \underline{\dim}\, J^l N \;\; \textit{for all} \;\; l \in \{0, \dots, L\}.$$

(3) **Socle layering:** $\operatorname{soc}_l M = \operatorname{ann}_M J^{l+1}$ *and* $\underline{\dim} \operatorname{soc}_l M = \underline{\dim} \bigoplus_{0 \le j \le l} \mathbb{S}^*_j(M)$*; in particular,*

$$\mathbb{S}^*(M) \le \mathbb{S}^*(N) \quad \iff \quad \underline{\dim} \operatorname{soc}_l M \le \underline{\dim} \operatorname{soc}_l N \;\; \textit{for all} \;\; l \in \{0, \dots, L\}.$$

(4) **Connection:** $J^l M \subseteq \operatorname{soc}_{L-l} M$*, and hence* $\bigoplus_{l \le j \le L} \mathbb{S}_j(M) \subseteq \bigoplus_{0 \le j \le L-l} \mathbb{S}^*_j(M)$.

We conclude this subsection with two further lemmas required later. Throughout, we equip the Cartesian product $\mathfrak{Seq}(\mathbf{d}) \times \mathfrak{Seq}(\mathbf{d})$ with the componentwise dominance order.

Lemma 2.4. *Suppose M is a Λ-module with dimension vector $\mathbf{d}$.*

(a) *First suppose that $\mathbb{S}(M) = (\mathbb{S}_0, \dots, \mathbb{S}_L)$. Then*
- $(\mathbb{S}_L, \mathbb{S}_{L-1}, \dots, \mathbb{S}_0) \le \mathbb{S}^*(M)$.
- *If $\mathbb{S}^*(M) = (\mathbb{S}_L, \dots, \mathbb{S}_0)$, then $\big(\mathbb{S}(M), \mathbb{S}^*(M)\big)$ is minimal in* **rad-soc**$(\mathbf{d})$.

(b) *Next suppose that $\mathbb{S}^*(M) = (\mathbb{S}^*_0, \dots, \mathbb{S}^*_L)$. Then*
- $(\mathbb{S}^*_L, \mathbb{S}^*_{L-1}, \dots, \mathbb{S}^*_0) \le \mathbb{S}(M)$.

- *If* $\mathbb{S}(M) = (\mathbb{S}^*_L, \dots, \mathbb{S}^*_0)$*, then* $\big(\mathbb{S}(M), \mathbb{S}^*(M)\big)$ *is minimal in* **rad-soc(d)**.

Proof. We verify (a), part (b) being dual. By Lemma 2.3, we have $J^{L-l}M \subseteq \operatorname{soc}_l M$ for $0 \le l \le L$. Thus the direct sum of the simple composition factors of $J^{L-l}M$ is contained in the direct sum of the simple composition factors of $\operatorname{soc}_l M$. Since the former direct sum is

$$\bigoplus_{0\le j\le l} J^{L-j}M/J^{L-j+1}M = \bigoplus_{0\le j\le l} \mathbb{S}_{L-j}$$

and the latter equals

$$\operatorname{soc}_0 M \oplus \bigoplus_{1\le j\le l} \operatorname{soc}_j M/\operatorname{soc}_{j+1} M = \bigoplus_{0\le j\le l} \mathbb{S}^*_j(M),$$

the first claim under (a) follows.

To justify the second, suppose $\mathbb{S}^*(M) = (\mathbb{S}_L, \dots, \mathbb{S}_0)$, and let N be a module with dimension vector $\mathbf{d}$ such that $\big(\mathbb{S}(N), \mathbb{S}^*(N)\big) \le \big(\mathbb{S}(M), \mathbb{S}^*(M)\big)$. In particular, $\mathbb{S}(N) \le \mathbb{S}(M)$. Hence $\operatorname{soc}_0 M = J^L M \subseteq J^L N \subseteq \operatorname{soc}_0 N$ by hypothesis and Lemma 2.3. Using the inequality of the second entries, we deduce $\mathbb{S}^*_0(M) = \operatorname{soc}_0 M = \operatorname{soc}_0 N = \mathbb{S}^*_0(N)$, whence also $J^L M = J^L N$. Analogously, one finds $\operatorname{soc}_1 M/\operatorname{soc}_0 M = J^{L-1}M/J^L M \subseteq J^{L-1}N/J^L N \subseteq \operatorname{soc}_1 N/\operatorname{soc}_0 N$, and a repeat of the initial argument shows that these inclusions are again equalities. An obvious induction thus yields equality of the pairs $\big(\mathbb{S}(M), \mathbb{S}^*(M)\big)$ and $\big(\mathbb{S}(N), \mathbb{S}^*(N)\big)$. $\square$

Lemma 2.5. *For* $M \in \Lambda$-mod *the following conditions are equivalent:*

(a) $\mathbb{S}^*(M) \ne \big(\mathbb{S}_L(M), \dots, \mathbb{S}_0(M)\big)$.

(b) *There exist an index* $\rho \in \{1, \dots, L\}$ *and an element* $x \in M \setminus J^\rho M$ *such that* $Jx \subseteq J^{\rho+1}M$.

(c) *There exists an index* $\rho \in \{0, \dots, L-1\}$ *such that* $J^\rho M/J^{\rho+2}M$ *has a simple direct summand.*

Proof. Suppose (a) holds, and let $m \ge 0$ be minimal with the property that

$$\big(\mathbb{S}^*_0(M), \dots, \mathbb{S}^*_m(M)\big) \ne \big(\mathbb{S}_L(M), \dots, \mathbb{S}_{L-m}(M)\big).$$

Since $J^{L-l}M \subseteq \operatorname{soc}_l M$ for $0 \le l \le L$ by Lemma 2.3, this amounts to $\operatorname{soc}_j M = J^{L-j}M$ for $j < m$ and $\operatorname{soc}_m M \supsetneqq J^{L-m}M$. Hence

$$\operatorname{soc}_m M/\operatorname{soc}_{m-1} M = \operatorname{soc}\big(M/J^{L-(m-1)}M\big) \supsetneqq J^{L-m}M/J^{L-m+1}M.$$

We infer that (b) is satisfied with $\rho = L - m$.

The implication "(b) $\Longrightarrow$ (a)" is proved analogously, and the equivalence of (b) and (c) is obvious. $\square$

2.C. Cornering the components of $\mathbf{Rep}_\mathbf{d}(\Lambda)$ via upper semicontinuous maps.

Definition 2.6. Suppose X is a topological space and $(\mathcal{A}, \leq)$ a poset. For $a \in \mathcal{A}$, we denote by $[a, \infty)$ the set $\{b \in \mathcal{A} \mid b \geq a\}$; the sets (a, ∞), $(-\infty, a]$ and $(-\infty, a)$ are defined analogously.

A map $f : X \longrightarrow \mathcal{A}$ is called *upper semicontinuous* if, for every element $a \in \mathcal{A}$, the pre-image of $[a, \infty)$ under f is closed in X.

The following module invariants are well-known to yield upper semicontinuous maps on $X = \mathbf{Rep_d}(\Lambda)$. They all take numerical values and have finite images, hence satisfy the hypotheses of the upcoming observation. For any fixed $N \in \Lambda$-mod, the maps $x \mapsto \dim \operatorname{Hom}_\Lambda(M_x, N)$ and $x \mapsto \dim \operatorname{Hom}_\Lambda(N, M_x)$, $x \mapsto \dim \operatorname{Ext}^1_\Lambda(M_x, N)$, and $x \mapsto \dim \operatorname{Ext}^1_\Lambda(N, M_x)$ are examples; for Ext^1, see [**6**]. Moreover, for any path p in $KQ \setminus I$, the map $x \mapsto \operatorname{nullity}_p M_x$ is upper semicontinuous; here $\operatorname{nullity}_p M_x$ is the nullity of the K-linear map $M_x \to M_x,\ m \mapsto p\,m$.

The next observation is pivotal in the present context.

Observation 2.7. *Let $\mathcal{A}$ be a poset, X a topological space, and $f : X \to \mathcal{A}$ an upper semicontinuous map whose image is well partially ordered* (meaning that $\operatorname{Im}(f)$ does not contain any infinite strictly descending chain and every nonempty subset has only finitely many minimal elements).

Then the pre-images $f^{-1}\big((-\infty, a)\big)$ and $f^{-1}\big((-\infty, a]\big)$ for $a \in \mathcal{A}$ are open in X. In particular, given any irreducible subset $\mathfrak{U}$ of X, the restriction of f to $\mathfrak{U}$ is generically constant, and the generic value of f on $\mathfrak{U}$ is

$$\min\{f(x) \mid x \in \mathfrak{U}\}.$$

Proof. We address openness of $f^{-1}\big((-\infty, a]\big)$. The case where $(-\infty, a]$ contains $\operatorname{Im}(f)$ is trivial. Otherwise, let $c_1, \dots, c_m$ be the minimal elements of the set $\operatorname{Im}(f) \setminus (-\infty, a]$; the number of such elements is finite by hypothesis. Then $f^{-1}\big((-\infty, a]\big)$ is the complement in $\mathbf{Rep_d}(\Lambda)$ of the finite union $\bigcup_{i \leq m} f^{-1}\big([c_i, \infty)\big)$ of closed sets.

Now let $\mathfrak{U}$ be an irreducible subset of X (in particular, $\mathfrak{U} \neq \varnothing$), and let $c_1, \dots, c_m$ be the distinct minimal elements in $f(\mathfrak{U})$. Then the sets $\mathfrak{U} \cap f^{-1}\big((-\infty, c_i]\big) = f^{-1}(c_i)$ are non-empty and open in $\mathfrak{U}$. Since they are pairwise disjoint, we conclude that $m = 1$. This makes c_1 the generic value of f on $\mathfrak{U}$. $\square$

Observation 2.7 entails that, for any noetherian topological space X and any minimal element $a \in \operatorname{Im}(f)$, the closure $\overline{f^{-1}(a)}$ is a finite union of irreducible components of X. This motivates the following terminology.

Definition 2.8. Let X be a noetherian topological space, $\mathcal{A}$ a poset, and $f : X \to \mathcal{A}$ an upper semicontinuous map. We say that f *detects an irreducible component $\mathcal{C}$ of X* in case the generic value of f on $\mathcal{C}$ is minimal in $\operatorname{Im}(f)$; equivalently, $\mathcal{C} \cap f^{-1}(a) \neq \varnothing$ for some minimal element $a \in \operatorname{Im}(f)$. Further, f is said to *separate irreducible components* in case $f^{-1}(a)$ is irreducible for every minimal element $a \in \operatorname{Im}(f)$.

We illustrate the resulting strategy of detecting and separating.

Example 2.9. Let Λ be an algebra with $J^2 = 0$ and $\mathbf{d}$ any dimension vector. Moreover, let $\mathcal{A}$ be the set of pairs of semisimple modules with dimension vectors $\le \mathbf{d}$. As was shown in [**3**], the function

$$f : \mathbf{Rep_d}(\Lambda) \to \mathcal{A}, \quad x \mapsto \big(M_x/JM_x, \operatorname{soc} M_x\big)$$

detects all irreducible components of $\mathbf{Rep_d}(\Lambda)$ and separates them. This allows to list them from Q in terms of the generic tops of their modules. On the other hand, it is readily seen that neither of the upper semicontinuous maps $x \mapsto M_x/JM_x$ or $x \mapsto \operatorname{soc} M_x$ by itself suffices for a classification of the components. $\square$

In many situations, the combination of the following two upper semicontinuous maps on $X = \mathbf{Rep_d}(\Lambda)$ will turn out to be particularly discerning with regard to irreducible components. We remark that the set $\mathfrak{Seq}(\mathbf{d})$ of semisimple sequences with dimension vector $\mathbf{d}$ is finite. In particular, it is well partially ordered by dominance, as is the Cartesian product $\mathfrak{Seq}(\mathbf{d}) \times \mathfrak{Seq}(\mathbf{d})$ under the componentwise dominance order.

Observation 2.10. *For any dimension vector* $\mathbf{d}$*, the map*

$$\mathbf{Rep_d}(\Lambda) \longrightarrow \mathfrak{Seq}_{\mathbf{d}}, \quad x \mapsto \mathbb{S}(M_x)$$

is upper semicontinuous. Its image coincides with the set of realizable semisimple sequences in $\mathfrak{Seq}_{\mathbf{d}}$.

Analogously, the map

$$\mathbf{Rep_d}(\Lambda) \longrightarrow \mathfrak{Seq}_{\mathbf{d}}, \quad x \mapsto \mathbb{S}^*(M_x)$$

is upper semicontinuous, and its image is the set of duals of those semisimple sequences which are realizable in the category of right Λ*-modules.*

Proof. It was proved in [**13**, Observation 2.11] that, for any semisimple sequence $\mathbb{S}$ with $\mathbf{Rep}\,\mathbb{S} \ne \varnothing$, the set $\bigcup_{\mathbb{S}' \ge \mathbb{S}} \mathbf{Rep}\,\mathbb{S}'$ is closed in $\mathbf{Rep_d}(\Lambda)$. Upper semi-continuity of socle layerings follows by duality. The final claim, regarding the image of $\mathbb{S}^*$, is due to the equality

$$\mathbb{S}^*(M) = \big(D(\mathbb{S}_0(D(M))), \cdots, D(\mathbb{S}_L(D(M)))\big). \quad \square$$

We combine the two maps of Observation 2.10 to an upper semicontinuous map

$$\Theta : \mathbf{Rep_d}(\Lambda) \longrightarrow \mathfrak{Seq}_{\mathbf{d}} \times \mathfrak{Seq}_{\mathbf{d}}, \quad x \mapsto \big(\mathbb{S}(M_x), \mathbb{S}^*(M_x)\big).$$

In light of Observation 2.7, we wish to identify the pairs $\big(\mathbb{S}, \mathbb{S}^*\big)$ which are minimal in the image of Θ, i.e., minimal in the set $\mathbf{rad\text{-}soc}(\mathbf{d}) = \{\big(\mathbb{S}(M), \mathbb{S}^*(M)\big) \mid M \in \Lambda\text{-mod}, \underline{\dim}\, M = \mathbf{d}\}$. This will permit us to gauge the potential of Θ towards detection and separation of components in the sense of Definition 2.8.

We will see in Theorem 3.1 below that, for any truncated path algebra Λ, the map Θ separates the irreducible components of the module varieties, but it need not detect them all (see Example 4.8 below). However, beyond the class of truncated path algebras, separation of irreducible components by Θ is not guaranteed either, as the following example shows.

Example 2.11. Let $\Delta = KQ/I$, where Q is the quiver $1 \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} 2 \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} 3$ and I is generated by the two paths $\beta_j\alpha_i$ for $i \neq j$. Moreover, take $\mathbf{d} = (1,1,1)$. Then $\mathbf{Rep_d}(\Delta)$ has two irreducible components determined by their generic modules $\Delta e_1/\Delta\alpha_1$ and $\Delta e_1/\Delta\alpha_2$, respectively. The map Θ detects, but does not separate them. On the other hand, the pair of path nullities, $\mathbf{Rep_d}(\Delta) \to (\mathbb{N}_0)^2$, $x \mapsto \big(\text{nullity}_{\beta_1\alpha_1} M_x, \text{nullity}_{\beta_2\alpha_2} M_x\big)$, obviously again upper semicontinuous, does detect *and* separate the components. $\square$

3. Specializing to truncated path algebras

From now on, we assume $\Lambda = KQ/I$ to be a truncated path algebra of Loewy length $L+1$, i.e., $I = \langle$the paths of length $L+1\rangle$.

3.A. Generic radical layerings determine the irreducible components of $\mathbf{Rep_d}(\Lambda)$.

Theorem 3.1. (see [**1**, Theorem 5.3]) *Each of the subvarieties $\mathbf{Rep}\,\mathbb{S}$ of $\mathbf{Rep_d}(\Lambda)$ is irreducible, unirational and smooth, and all irreducible components of $\mathbf{Rep_d}(\Lambda)$ are among the closures $\overline{\mathbf{Rep}\,\mathbb{S}}$, where $\mathbb{S}$ traces the semisimple sequences with dimension vector $\mathbf{d}$. In particular, the map $\mathbf{Rep_d}(\Lambda) \to \mathfrak{Seq}(\mathbf{d})$, $x \mapsto \mathbb{S}(M_x)$ separates irreducible components.* $\square$

Naturally, in the process of sifting out the semisimple sequences $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ which give rise to the irreducible components of $\mathbf{Rep_d}(\Lambda)$, only the realizable ones are in the running. These are singled out by the following criterion; it may be checked by mere inspection of the quiver. To conveniently formulate it, we let P_l be a projective cover of $\mathbb{S}_l$ and denote by $\mathbf{B}$ the adjacency matrix of the quiver Q, i.e., $\mathbf{B}_{ij}$ is the number of arrows from e_i to e_j. Then the first radical layer of P_0, that is, the semisimple module $\mathbb{S}_1(P_0) = JP_0/J^2P_0$, clearly has dimension vector $\underline{\dim}\,\mathbb{S}_1(P_0) = \underline{\dim}\,\mathbb{S}_0 \cdot \mathbf{B}$ (this is actually true for any path algebra modulo an admissible ideal). Moreover, one notes that, for a truncated path algebra Λ, the $(l+1)$-th radical layer $\mathbb{S}_{l+1}(P_0)$ of P_0 coincides with the first radical layer $\mathbb{S}_1(P_l)$ of P_l, whence $\underline{\dim}\,\mathbb{S}_{l+1}(P_0) = \underline{\dim}\,\mathbb{S}_l \cdot \mathbf{B} = \underline{\dim}\,\mathbb{S}_0 \cdot \mathbf{B}^{l+1}$.

Realizability Criterion 3.2.. *We retain the above notation. For a semisimple sequence $\mathbb{S} = (\mathbb{S}_0, \mathbb{S}_1, \dots, \mathbb{S}_L)$ the following conditions are equivalent:*

(1) *$\mathbb{S}$ is realizable.*
(2) *For each $l \in \{0, \dots, L-1\}$, the two-term sequence $(\mathbb{S}_l, \mathbb{S}_{l+1})$ is realizable.*
(3) *$\underline{\dim}\,\mathbb{S}_{l+1} \le \underline{\dim}\,\mathbb{S}_1(P_l)$ for $l \in \{0, \dots, L-1\}$, i.e., $\underline{\dim}\,\mathbb{S}_{l+1} \le \underline{\dim}\,\mathbb{S}_l \cdot \mathbf{B}$.*

Proof. "(3) $\implies$ (1)": Suppose that $\mathbb{S}$ satisfies (3), and let $\underline{\dim}\,\mathbb{S}_l = (t_{l1}, \dots, t_{ln})$ for $0 \le l \le L$. Then $t_{l+1,j} \le \sum_{1 \le k \le n} t_{lk}\mathbf{B}_{kj}$ for all eligible l and j. We first break up $\mathbb{S}$ into a direct sum of smaller semisimple sequences, $\mathbb{S} = \bigoplus_{1 \le i \le n, 1 \le r \le t_{0i}} \mathbb{S}^{(i,r)}$, such that the top $\mathbb{S}_0^{(i,r)}$ is S_i, and each $\mathbb{S}^{(i,r)}$ satisfies (3). For that purpose, we will inductively choose dimension vectors $\mathbf{d}_l^{(i,r)} = (t_{l1}^{(i,r)}, \dots, t_{ln}^{(i,r)})$, for $1 \le i \le n$, $1 \le r \le t_{0i}$ and $0 \le l \le L$, with the property that

$$\mathbf{d}_0^{(i,r)} = \underline{\dim}\,S_i \quad \text{and} \quad \sum_{i,r} \mathbf{d}_l^{(i,r)} = \underline{\dim}\,\mathbb{S}_l, \quad \text{while} \quad t_{l+1,j}^{(i,r)} \le \sum_k t_{lk}^{(i,r)}\mathbf{B}_{kj}$$

for all eligible i, r, l, j. Assuming that the $\mathbf{d}_l^{(i,r)}$ have been chosen as required for some $l < L$, we have

$$\mathfrak{t}_{l+1,j} \le \sum_{1 \le k \le n} t_{lk} \mathbf{B}_{kj} \;=\; \sum_{i \le n,\, r \le t_{0i}} \; \sum_{1 \le k \le n} t_{lk}^{(i,r)} \mathbf{B}_{kj}.$$

This allows us to choose nonnegative integers $t_{l+1,j}^{(i,r)} \le \sum_{1 \le k \le n} t_{lk}^{(i,r)} \mathbf{B}_{kj}$ with the property that $t_{l+1,j} = \sum_{i \le n,\, r \le t_{0i}} t_{l+1,j}^{(i,r)}$ for all j. Now we set $d_{l+1}^{(i,r)} = (t_{l+1,1}^{(i,r)}, \dots, t_{l+1,n}^{(i,r)})$, which completes the induction. Let $\mathbb{S}^{(i,r)} = (\mathbb{S}_0^{(i,r)}, \dots, \mathbb{S}_L^{(i,r)})$ with $\underline{\dim}\, \mathbb{S}_l^{(i,r)} = \mathbf{d}_l^{(i,r)}$ for all l; in particular, $\mathbb{S}_0^{(i,r)} = S_i$. By construction, each of the sequences satisfies (3), and $\mathbb{S}$ has the postulated direct sum decomposition.

If, for each choice of (i, r), we can find a module $M^{(i,r)}$ with radical layering $\mathbb{S}^{(i,r)}$, the direct sum $\bigoplus_{i,r} M^{(i,r)}$ will have radical layering $\mathbb{S}$, thus revealing realizability of $\mathbb{S}$. Therefore the proof is reduced to the case where $\mathbb{S}_0$ is simple, say $\mathbb{S}_0 = S_i$.

The graph of the Λ-module Λe_i is a tree $\mathcal{T}_i$ having root e_i, the rooted branches being in bijective correspondence with those paths of length $\le L$ in Q which start in e_i; in this correspondence, the number of rooted branches of $\mathcal{T}_i$ that have length l and end in a vertex labeled by e_j equals the multiplicity $\dim e_j\, \mathbb{S}_l(\Lambda e_i)$ of the simple module S_j in $J^l e_i / J^{l+1} e_i$. Observe that $\dim e_j\, \mathbb{S}_l(\Lambda e_i) = (\mathbf{B}^l)_{i,j}$ for all j and l. Due to the inequality $t_{l+1,j} \le \sum_k t_{lk} \mathbf{B}_{kj}$, an easy induction on L yields a rooted subtree $\mathcal{T}_i'$ of $\mathcal{T}_i$ with the following property: For each l and j, the number of those rooted branches of $\mathcal{T}_i'$ which have length l and end in e_j equals t_{lj}. Now take M to be the module $\Lambda e_i / U$, where U is the sum of the cyclic submodules Λq with q tracing the paths in Q which start in e_i and are minimal with respect to not being among the rooted branches of $\mathcal{T}_i'$. One readily checks that $\mathbb{S}(M) = \mathbb{S}$.

We leave the proof of the remaining implications to the reader. $\square$

We contrast Theorem 3.1 and Criterion 3.2 with the situation of a non-truncated algebra.

Example 3.3. Consider the algebra $\Delta = KQ / \langle \beta_i \alpha_j \mid i \ne j \rangle$ of Example 2.11. If $\mathbb{S} = (S_1, S_2, S_3)$, then $\mathbf{Rep}\, \mathbb{S}$ is reducible, as we saw earlier. Moreover, the sequence $\mathbb{S}' = (S_1, S_2, S_3^2)$ fails to be realizable, even though it satisfies Conditions (2), (3) of Criterion 3.2. $\square$

3.B. Skeleta and generic minimal projective presentations over truncated path algebras.

We recall some concepts which were defined for and applied to arbitrary basic finite dimensional algebras (see, e.g., [**1**] or [**13**]). Restriction to the truncated case simplifies the picture, since there is a well-defined notion of path length in Λ. We thus pare down the general definitions and results so as to take advantage of the current situation.

Roughly, a skeleton of a Λ-module M is a path basis for M which lives in a fixed projective cover P of M, displays the radical layering of M in terms of path lengths, and is "shared" by all modules in a dense open subset of the irreducible variety $\mathbf{Rep}\, \mathbb{S}(M)$. In particular, the set of skeleta of a module is a generic attribute of the modules in any irreducible component of $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$.

Definition 3.4. Let $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ be a semisimple sequence with dimension vector $\mathbf{d}$ and $\dim \mathbb{S}_0 = t$. Fix a projective cover $P = \bigoplus_{1 \le r \le t} \Lambda z_r$ of $\mathbb{S}_0$, where $z_1, \dots, z_t$ is a full sequence of top elements of P. (We point out that, typically, P is a proper direct summand of the projective module $\mathbf{P}$ on which we based the definition of $\mathrm{GRASS}_{\mathbf{d}}(\Lambda)$ in Section 1; in fact, $\underline{\dim}\, \mathbf{P}/J\mathbf{P} = \mathbf{d}$.) Any *nonzero* element of the form $pz_r \in P$, where p is a path in Q, is called a *path in* P, and we set $\mathrm{length}(pz_r) = \mathrm{length}(p)$. Note that this definition is unambiguous. The paths of length 0 in P are precisely those of the form $z_r = e(r)z_r$, $1 \le r \le t$, where $e(r)$ is the primitive idempotent that norms z_r. The *endpoint*, $\mathrm{end}(pz_r)$, of a path pz_r is that of p.

(a) An (*abstract*) *skeleton with radical layering* $\mathbb{S}$ is a set σ of paths in P such that

• σ is closed under initial subpaths, meaning: whenever $p = p_2p_1$ for paths p_i in Q such that $p_2p_1z_r \in \sigma$, it follows that $p_1z_r \in \sigma$.

• For any $l \in \{0, \dots, L\}$ and $i \in \{1, \dots, n\}$, the number of those paths of length l in σ which end in e_i equals the multiplicity of S_i in $\mathbb{S}_l$.

(b) A *skeleton* of $M \in \Lambda$-mod is a skeleton σ with radical layering $\mathbb{S}(M)$ such that $M \cong P/C$ and $\{\mathbf{p} + C \mid \mathbf{p} = pz_r \in \sigma\}$ is a basis for P/C.

(c) We say that a path $\mathbf{q} = qz_r \in P$ is σ*-critical* if $\mathbf{q} \notin \sigma$, while all proper initial subpaths of $\mathbf{q}$ belong to σ. For any σ-critical path $\mathbf{q}$, let

$$\sigma(\mathbf{q}) := \{\mathbf{p} = pz_s \in \sigma \mid \mathrm{length}(\mathbf{p}) \ge \mathrm{length}(\mathbf{q}) \text{ and } \mathrm{end}(\mathbf{p}) = \mathrm{end}(\mathbf{q})\}.$$

Clearly, the set of abstract skeleta with any fixed radical layering is finite, and every Λ-module has at least one skeleton. Conversely, each abstract skeleton arises as the skeleton of a module; to see this, consult the proof of Criterion 3.2. In light of their graphical rendering, skeleta provide a visually suggestive mode of communicating *realizable* semisimple sequences over Λ: Indeed, $\mathbb{S}$ is realizable if and only if there exists an abstract skeleton with radical layering $\mathbb{S}$.

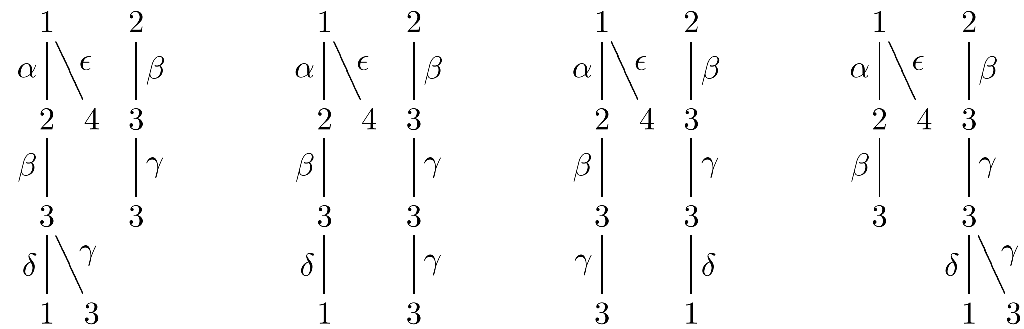
Skeleta may be thought of as layered and labeled graphs. This is illustrated by

Example 3.5. Let Λ be the truncated path algebra of Loewy length 4 over K based on the quiver

$$\begin{tikzcd} 4 & 1 \arrow[l, "\epsilon"'] \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] & 3 \arrow[ll, bend left, "\delta"] \arrow[loop right, "\gamma"] \end{tikzcd}$$

and let $\mathbb{S} = (S_1 \oplus S_2, S_2 \oplus S_3 \oplus S_4, S_3^2, S_1 \oplus S_3)$. Here $P = \Lambda z_1 \oplus \Lambda z_2$ with $z_i = e_i z_i$ for

$i = 1, 2$. There are precisely 4 skeleta with radical layering $\mathbb{S}$, namely



The left-hand pair σ of trees, for instance, stands for the skeleton consisting of the 9 rooted edge paths on display; these include the paths of length zero z_1 and z_2 communicated by the roots of the trees, labeled e_1 and e_2 (alias 1 and 2). Moreover, $\mathbf{q} = \gamma^2 \beta z_2$ is a σ-critical path with $\sigma(\mathbf{q}) = \{\gamma\beta\alpha z_1\}$. $\square$

The following theorem follows from [**1**, Section 5]. It tells us that, identifying the generic radical layerings of the irreducible components of $\mathbf{Rep_d}(\Lambda)$ provides us with immediate access to a significant amount of generic information about the modules in these components. Keep in mind that Λ is a truncated path algebra.

Theorem 3.6. Generic minimal projective presentations. *Let $\mathbb{S}$ be a realizable semisimple sequence and σ any abstract skeleton with radical layering $\mathbb{S}$. Again, P is a projective cover of $\mathbb{S}_0$ and $z_1, \dots, z_t$ a distinguished sequence of top elements of P.*

Generically, the modules in $\overline{\mathbf{Rep}\,\mathbb{S}}$ then have skeleton σ. Moreover, the modules with skeleton σ are precisely those Λ-modules which have a presentation of the following format: $M \cong P/U(\mathbf{c})$, where $U(\mathbf{c})$ is the submodule of P which is generated by the differences

$$\mathbf{q} \; - \sum_{\mathbf{p} \in \sigma(\mathbf{q})} c_{\mathbf{q},\mathbf{p}}\, \mathbf{p} \qquad \textit{for } \; \mathbf{q} = q z_r \;\; \sigma\textit{-critical},$$

where $\mathbf{c} = (c_{\mathbf{q},\mathbf{p}})$ is an arbitrary family of scalars in K.

In particular: Generically, the modules in $\overline{\mathbf{Rep}\,\mathbb{S}}$ have minimal projective presentations of the form displayed above.

Proof. For the first claim, see [**1**, Theorem 5.3]. For the second part, combine [**1**, remarks preceding Theorem 3.8] with [**1**, Theorem 5.3]. For the final assertion, observe that $\mathbf{Rep}\,\mathbb{S}$, being locally closed, is open in its closure, and that the set of those points in $\mathbf{Rep}\,\mathbb{S}$ which represent modules with skeleton σ is, in turn, open in $\mathbf{Rep}\,\mathbb{S}$; for the latter fact, see [**1**, Observation 3.5]. In all of these citations, [**1**] proves the pertinent statements for the varieties of Grassmann type described in Section 2.A above. Use Proposition 2.1 for a translation into the affine setting $\mathbf{Rep_d}(\Lambda)$. $\square$

The final assertion of the upcoming corollary addresses generic modules. For a definition, we refer to [**1**, Section 4]. Here, this concept plays only a peripheral role.

Corollary 3.7. *Again let* $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ *be a realizable semisimple sequence, and suppose that* $0 \le \rho < \tau \le L$. *Then there are dense open subsets* $\mathfrak{U} \subseteq \mathbf{Rep}\,\mathbb{S}$ *and* $\mathfrak{V} \subseteq \mathbf{Rep}(\mathbb{S}_\rho, \dots, \mathbb{S}_\tau)$ *such that the set of (isomorphism classes of) modules in* $\mathfrak{V}$ *coincides with the set of subfactors* $J^\rho M / J^{\tau+1} M$ *of the modules* M *in* $\mathfrak{U}$. *In other words, generically, the modules in* $\mathbf{Rep}(\mathbb{S}_\rho, \dots, \mathbb{S}_\tau)$ *coincide with the specified subfactors of the modules in* $\mathbf{Rep}\,\mathbb{S}$

In particular: If G *is a generic module for* $\mathbf{Rep}\,\mathbb{S}$, *then* $J^\rho G / J^{\tau+1} G$ *is a generic module for* $\mathbf{Rep}(\mathbb{S}_\rho, \dots, \mathbb{S}_\tau)$. $\square$

3.C. A brief return to socle layerings.

We prepare for the proof of the Main Theorem with a final lemma.

Lemma 3.8. *Let* $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ *be a realizable semisimple sequence, and denote by* $E(\mathbb{S}_l)$ *the injective envelope of* $\mathbb{S}_l$. *Suppose that, for each* $l \in \{1, \dots, L\}$, *the semisimple module* $\mathbb{S}_{l-1}$ *embeds into the first socle layer* $\mathbb{S}_1^*\big(E(\mathbb{S}_l)\big) = \operatorname{soc}\big(E(\mathbb{S}_l)/\mathbb{S}_l\big)$ *of* $E(\mathbb{S}_l)$. *Then the generic socle layering of the modules in* $\mathbf{Rep}\,\mathbb{S}$ *is* $(\mathbb{S}_L, \dots, \mathbb{S}_0)$, *and* $\overline{\mathbf{Rep}\,\mathbb{S}}$ *is an irreducible component of* $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$.

Proof. We start by proving that the generic socle layering of $\mathbf{Rep}\,\mathbb{S}$ is as postulated. By Lemma 2.5, it suffices to show that, generically, the modules $M \in \mathbf{Rep}\,\mathbb{S}$ satisfy the following condition (†): For each $l \in \{1, \dots, L\}$, the subquotient $J^{l-1}M/J^{l+1}M$ in $\mathbf{Rep}(\mathbb{S}_{l-1}, \mathbb{S}_l)$ is free of simple direct summands.

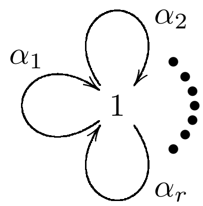
In view of the realizability of $\mathbb{S}$, there is a module U with $\mathbb{S}(U) = (\mathbb{S}_{l-1}, \mathbb{S}_l)$. This means $\operatorname{soc} U = X \oplus JU$ and $U = X \oplus Y$ for some submodules X, Y with $JY = JU = \mathbb{S}_l$. In particular, $\mathbb{S}_l \subseteq Y \subseteq E(\mathbb{S}_l)$, up to isomorphism. By hypothesis, $X \oplus (Y/JU) = U/JU$ embeds into $\mathbb{S}_1^*\big(E(\mathbb{S}_l)\big)$, so X embeds into a direct complement of $Y/\mathbb{S}_l$ in $\mathbb{S}_1^*\big(E(\mathbb{S}_l)\big)$. This provides us with a submodule V of $E(\mathbb{S}_l)$ such that $V \supseteq \mathbb{S}_l$ and $V/\mathbb{S}_l \cong \mathbb{S}_{l-1}$. We infer $JV = \mathbb{S}_l$ and conclude $\mathbb{S}(V) = (\mathbb{S}_{l-1}, \mathbb{S}_l)$.

This makes $\mathbb{S}_l$ the unique smallest semisimple module arising as the socle of a module in $\mathbf{Rep}(\mathbb{S}_{l-1}, \mathbb{S}_l)$. Using Observations 2.7 and 2.10, we deduce that, generically, the modules in $\mathbf{Rep}(\mathbb{S}_{l-1}, \mathbb{S}_l)$ have socle $\mathbb{S}_l$ and consequently are free of simple direct summands. Combine with Corollary 3.7 to conclude that, generically, the specified subquotients $J^{l-1}M/J^{l+1}M$ satisfy (†). This proves the first claim.

In light of Lemma 2.4 and Observations 2.7 and 2.10, the final assertion follows. $\square$

4. The local case. Consequences of the Main Theorem and examples

Barring the final example (4.8), we consider only *local* truncated path algebras in this section. Throughout, Λ denotes the local truncated path algebra of Loewy length $L+1$ over K with $\dim_K J/J^2 = r$. Thus Λ is based on the following quiver:



In the following, we assume the number r of loops to be at least 2. The Main Theorem is stated in the introduction.

We start by pointing to the fact that knowledge of the generic radical layerings of the irreducible components of $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$ yields explicit (if, a priori, not very convenient) generic minimal projective presentations of the modules in the components. This is substantiated by Theorem 3.6.

The next consequence of the Main Theorem allows us to garner further generic information about the irreducible components of $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$ by passing back and forth among $\mathbf{d}$-dimensional semisimple sequences $\mathbb{S}$ and semisimple sequences with smaller dimension vectors and Loewy lengths; see Corollary 3.7 in this connection.

Corollary 4.1. *Suppose $d > L+1$, and let $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ be a realizable semisimple sequence with $\dim \mathbb{S} = d$. Then $\mathbb{S}$ is the generic radical layering of an irreducible component of $\mathbf{Rep}_d(\Lambda)$ if and only if, for each $l \in \{1, \dots, L\}$, the two-term sequence $(\mathbb{S}_{l-1}, \mathbb{S}_l)$ is the generic radical layering of an irreducible component of $\mathbf{Rep}_{d^{(l)}}(\Lambda/J^2)$, where $d^{(l)} = \dim \mathbb{S}_{l-1} + \dim \mathbb{S}_l$.* $\square$

Note that, for $d \le L+1$, the variety $\mathbf{Rep}_d(\Lambda)$ is irreducible by the Main Theorem and has generic radical layering $\mathbb{S}$ consisting of simple entries. In particular, $(\mathbb{S}_{l-1}, \mathbb{S}_l)$ is the generic radical layering of $\mathbf{Rep}_2(\Lambda/J^2)$ whenever $\mathbb{S}_l \neq 0$. Clearly, Corollary 4.1 extends to arbitrary segments of $\mathbb{S}$. If $\dim \mathbb{S} > L+1$, this amounts to the following: $\mathbb{S}$ is the generic radical layering of an irreducible component of $\mathbf{Rep}_d(\Lambda)$ if and only if, for any choice of $\rho < \tau$ in $\{0, \dots, L\}$, the trimmed sequence $(\mathbb{S}_\rho, \dots, \mathbb{S}_\tau)$ is the generic radical layering of an irreducible component of the variety of $(\Lambda/J^{\tau-\rho+1})$-modules of dimension $\sum_{\rho \le l \le \tau} \dim \mathbb{S}_l$.

Corollary 4.2. *Suppose $d \ge L+1$, and let $\mathcal{C}$ be an irreducible component of $\mathbf{Rep}_d(\Lambda)$. Then the modules M in $\mathcal{C}$ generically satisfy the following condition: $J^L x \neq 0$ whenever $x \in M \setminus JM$.*

In particular: There exists a dense open subset $\mathfrak{U} \subseteq \mathcal{C}$ such that every module in $\mathfrak{U}$ has a decomposition into indecomposable direct summands <u>all</u> of which have Loewy length $L+1$.

Proof. Suppose $\mathcal{C} = \overline{\mathbf{Rep}\,\mathbb{S}}$, and let M be a module in $\mathbf{Rep}\,\mathbb{S}$ whose socle layering is the generic one, i.e.,

$$(\dagger) \qquad \mathbb{S}^*(M) = \big(\mathbb{S}_L, \dots, \mathbb{S}_0\big)$$

by the Main Theorem; to verify this socle layering, use part (I) of the theorem for $d > L+1$, and part (II) for $d = L+1$. Moreover, suppose $x \in M \setminus JM$. We derive the inequality $J^L x \neq 0$ from $(\dagger)$ by induction on $L \ge 1$. The case $L = 1$ is settled by Lemma 2.5. Indeed, this lemma guarantees that M has no simple direct summand. For the induction step, let $L > 1$, set $\overline{M} = M/J^L M$, and write $\overline{x} = x + J^L M$. Observe that $\overline{d} = \dim \overline{M} \geq L$, since $\mathbb{S}_{L-1}(\overline{M}) \neq 0$. Clearly, the radical layering of $\overline{M}$ is $(\mathbb{S}_0, \dots, \mathbb{S}_{L-1})$, and the socle layering of $\overline{M}$ equals $(\mathbb{S}_{L-1}, \dots, \mathbb{S}_0)$ by (†), because $\overline{M} = M/\operatorname{soc} M$. Hence $J^{L-1}\overline{x} \neq 0$ by the induction hypothesis, which amounts to $J^{L-1}x \not\subseteq J^L M$; in other words, $J^{L-1}x \subseteq J^{L-1}M \setminus J^L M$. Once more invoking Lemma 2.5, we find that $J^{L-1}M$ is free of simple direct summands, whence we conclude that $J(J^{L-1}x) \neq 0$ as required. $\square$

It is easy to see that Corollary 4.2 has no analogue for general truncated path algebras. The preceding corollaries, in turn, have an interesting consequence. Namely, in many cases, the test for generic indecomposability of the modules in an irreducible component of $\mathbf{Rep}_d(\Lambda)$ may be played back to the generalized Kronecker quiver $\widehat{Q}$:

$$\begin{tikzcd} 1 \arrow[r, bend left=50, "\beta_1"] \arrow[r, "\beta_2"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=50, "\beta_r"'] & 2 \end{tikzcd}$$

The Schur roots of $\widehat{Q}$, i.e., the dimension vectors (d_1, d_2) with the property that the modules in $\mathbf{Rep}_{(d_1,d_2)} K\widehat{Q}$ are generically indecomposable, were already determined by Kac in [**16**, Section 2.6]. Observe that, whenever $d \leq L + 1$, the modules in the irreducible variety $\mathbf{Rep}_d(\Lambda)$ are generically indecomposable by the Main Theorem. Beyond that we obtain:

Corollary 4.3. *Suppose $d > L+1$, and let $\mathbb{S}$ be the generic radical layering of the modules in an irreducible component $\mathcal{C}$ of $\mathbf{Rep}_d(\Lambda)$. A sufficient condition for the modules in $\mathcal{C}$ to be generically indecomposable is as follows: There exists an index $l \in \{1, \dots, L\}$ such that the pair $(\dim \mathbb{S}_{l-1}, \dim \mathbb{S}_l)$ is a Schur root of the generalized Kronecker quiver $\widehat{Q}$.*

Moreover: In case at least one of the varieties $\mathbf{Rep}_{(\dim \mathbb{S}_{l-1}, \dim \mathbb{S}_l)} K\widehat{Q}$ contains infinitely many orbits of maximal dimension under the action of the pertinent general linear group, $\mathcal{C}$ contains infinitely many GL_d-orbits of maximal dimension.

Proof. Suppose that the modules in $\mathbf{Rep}_d(\Lambda)$ are generically decomposable. In other words, there exists a dense open subset $\mathfrak{U} \subseteq \mathcal{C}$ such that every module M in $\mathfrak{U}$ decomposes in the form $M = M_1 \oplus M_2$ with $\dim M_i$ fixed and nonzero. Generically, the modules M in $\mathfrak{U}$ then satisfy $J^L M_i \neq 0$ for $i = 1, 2$ by Corollary 4.2. In particular, we find that, for each $l \in \{1, \dots, L\}$, the quotient $J^{l-1}M/J^{l+1}M$ is in turn decomposable. By Corollaries 3.7 and 4.1, these quotients trace, for each l, a dense open subset of an irreducible component $\mathcal{C}_l$ of $\mathbf{Rep}_{d^{(l)}}(\Lambda/J^2)$, where $d^{(l)} = \dim \mathbb{S}_{l-1} + \dim \mathbb{S}_l$. Now [**3**, Theorem 5.6(b)] guarantees that the modules in $\mathbf{Rep}_{(\dim \mathbb{S}_{l-1}, \dim \mathbb{S}_l)} K\widehat{Q}$ are decomposable for $1 \leq l \leq L$; in other words, none of the dimension vectors $(\dim \mathbb{S}_{l-1}, \dim \mathbb{S}_l)$ is a Schur root of $\widehat{Q}$.

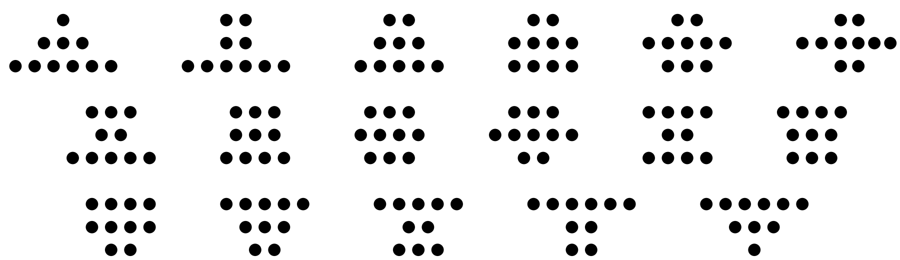
The final assertion follows from [**3**, Theorem 5.6(c)] by analogous reasoning. $\square$

The sufficient condition for generic indecomposability given in Corollary 4.3 fails to be necessary in general. Failure may even occur for $r = 2$:

Example 4.4. Let Λ be the local truncated $\mathbb{C}$-algebra of Loewy length 3 based on the quiver with 2 loops, and denote by S the unique simple in Λ-mod. For $d = 6$, the semisimple sequence $\mathbb{S} = (S^2, S^2, S^2)$ is the generic radical layering of an irreducible component of $\mathbf{Rep}_d(\Lambda)$, but $(2,2)$ fails to be a Schur root of the Kronecker algebra with two arrows. On the other hand, the modules in $\overline{\mathbf{Rep}\,\mathbb{S}}$ are generically indecomposable; indeed, using Theorem 3.6, one computes that, generically, the endomorphism rings of the modules in $\mathbf{Rep}\,\mathbb{S}$ have top $\mathbb{C}$. $\square$

We illustrate the Main Theorem and Corollary 4.3.

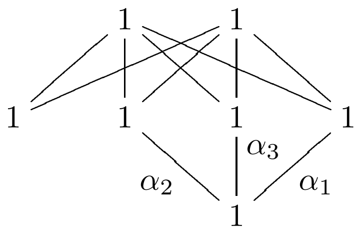
Example 4.5. Let Λ be the local truncated path algebra with $r = 3 = L + 1$, and $d = 10$. Then $\mathbf{Rep}_d(\Lambda)$ has precisely 17 irreducible components. Indeed, the eligible generic radical layerings are readily listed via Criterion (2) of the Main Theorem. They are displayed below, the bullets indicating the dimensions of the layers.



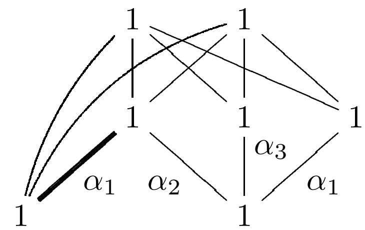
All of the components parametrize generically indecomposable modules and have infinitely many GL_d-orbits of maximal dimension; and in all but one of the cases (the last radical layering in the first row), reference to the generalized Kronecker algebra with 3 arrows yields these findings. For instance, the second of the irreducible components of $\mathbf{Rep}_{10}(\Lambda)$ is the closure $\mathcal{C}$ of $\mathbf{Rep}\,\mathbb{S}$, where $\mathbb{S} = (S^2, S^2, S^6)$. In this case $(\dim \mathbb{S}_0, \dim \mathbb{S}_1) = (2,2)$ is a Schur root of the Kronecker quiver with three arrows; see [**16**, Theorem 4(a)]. That $\mathcal{C}$ does not contain a dense orbit can be gleaned from [**3**, Theorem 5.6 (c)]. For the "exceptional" radical layering, one resorts to the generic presentation given in Theorem 3.6, supplemented by suitable polynomials which fail to vanish on the considered tuples $\mathbf{c}$ of scalars, so as to ensure indecomposability of the corresponding modules $P/U(\mathbf{c})$. $\square$

Next we provide simple instances of the two different deformation strategies underlying the proof of the Main Theorem. They facilitate visualization of the argument across technical hurdles.

Example 4.6 illustrating the proof of Lemma 5.2. Let Λ be the local truncated path algebra with $r = 3 = L + 1$. Any Λ-module M with a graph as shown on the left below deforms to a module $\widetilde{M}$ with a graph as shown on the right. By this we mean: M belongs to the closure $\overline{\mathbf{Rep}\,\mathbb{S}(\widetilde{M})}$.

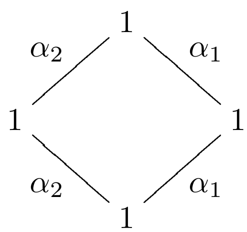


deforms to

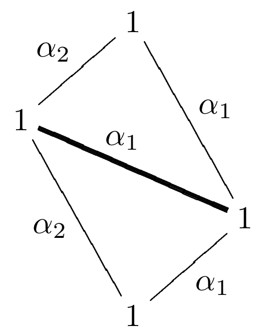


$\square$

Example 4.7 illustrating the first step of the proof of the Main Theorem. Let Λ be the local truncated path algebra with $r = 3 = L$. Any Λ-module M with a graph as shown on the left below deforms to a Λ-module $\widetilde{M}$ with a graph as shown on the right.



deforms to



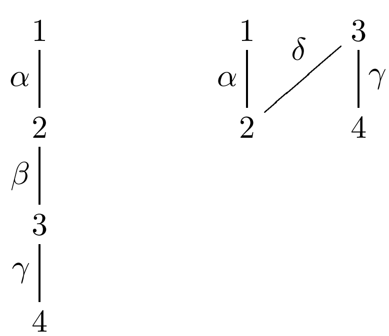
$\square$

As mentioned in the introduction, for nonlocal truncated path algebras of Loewy length ≥ 4, the upper semicontinuous map $(\mathbb{S}, \mathbb{S}^*)$ may fall short of detecting all irreducible components of the module varieties.

Example 4.8. Let $\Delta = KQ/\langle \text{all paths of length } 4\rangle$, where Q is the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, bend left, "\beta"] & 3 \arrow[l, bend left, "\delta"] \arrow[r, "\gamma"] & 4 \end{tikzcd}$$

For $\mathbf{d} = (1, 1, 1, 1)$, the variety $\mathbf{Rep}_{\mathbf{d}}(\Delta)$ consists of two irreducible components cut out by the following generic radical layerings: $\mathbb{S} = (S_1, S_2, S_3, S_4)$ and $\widetilde{\mathbb{S}} = (S_1 \oplus S_3, S_2 \oplus S_4, 0, 0)$. Both of the varieties $\mathbf{Rep}\,\mathbb{S}$ and $\mathbf{Rep}\,\widetilde{\mathbb{S}}$ contain dense orbits, namely those corresponding to the modules G and $\widetilde{G}$ determined by the following graphs, respectively; indeed, both G and $\widetilde{G}$ have only trivial self-extensions, whence their $\mathrm{GL}_{\mathbf{d}}$-orbits are open in $\mathbf{Rep}_{\mathbf{d}}(\Delta)$.



Therefore the generic socle layerings of $\mathbf{Rep}\,\mathbb{S}$ and $\mathbf{Rep}\,\widetilde{\mathbb{S}}$ are

$$\mathbb{S}^* = \mathbb{S}^*(G) = (S_4, S_3, S_2, S_1) \quad \text{and} \quad \widetilde{\mathbb{S}}^* = \mathbb{S}^*(\widetilde{G}) = (S_2 \oplus S_4, S_1 \oplus S_3, 0, 0),$$

respectively, which shows $(\mathbb{S}, \mathbb{S}^*) < (\widetilde{\mathbb{S}}, \widetilde{\mathbb{S}}^*)$. On the other hand, $\delta \cdot G = 0$ while $\delta \cdot \widetilde{G} \neq 0$, whence the corresponding generic triples

$$\big(\mathbb{S}(G), \mathbb{S}^*(G), \mathrm{nullity}_\delta(G)\big) \quad \text{and} \quad \big(\mathbb{S}(\widetilde{G}), \mathbb{S}^*(\widetilde{G}), \mathrm{nullity}_\delta(\widetilde{G})\big)$$

are not comparable; here $\mathrm{nullity}_\delta\, G = \dim \mathrm{ann}_G(\delta)$ as in the remarks preceding Observation 2.7. In fact, $\overline{\mathbf{Rep}\,\widetilde{\mathbb{S}}}$ is also an irreducible component of $\mathbf{Rep}_{\mathbf{d}}(\Delta)$. It is not detected by the map Θ alone. $\square$

5. Proof of the Main Theorem

Throughout this section, let Λ be as in the hypothesis of the Main Theorem, that is, $\Lambda = KQ/\langle \text{all paths of length } L+1 \rangle$, where Q has a single vertex and Q_1 consists of $r \geq 2$ loops, say $Q_1 = \{\alpha_1, \dots, \alpha_r\}$. The unique (up to isomorphism) simple left Λ-module is denoted by S. Further, $Q_{>0}$ will stand for the set of paths of positive length in Q. Given any family $(f_\alpha)_{\alpha \in Q_1}$ of K-endomorphisms of K^d, the following notation will be convenient: whenever $p = \alpha_{i_l} \cdots \alpha_{i_1}$ is a path in $Q_{>0}$, we let f_p be the corresponding composition of maps f_{α_j}.

We will repeatedly use the following specialization of Criterion 3.2 to the local case: Namely, a semisimple sequence $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ is realizable if and only if $\dim \mathbb{S}_l \leq r \cdot \dim \mathbb{S}_{l-1}$ for $1 \leq l \leq L$.

The initial lemma is our basic tool for constructing module deformations with prescribed radical layerings.

Lemma 5.1. Constructing Λ-module structures on graded vector spaces. *We refer to the above notation. Suppose that $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ is a d-dimensional semisimple sequence. Fix a direct sum decomposition of K^d of the form*

$$K^d = \bigoplus_{0 \leq l \leq L} K_l, \quad \textit{where} \quad \dim K_l = \dim \mathbb{S}_l \quad \textit{for } l \leq L,$$

and assume that $(f_\alpha)_{\alpha \in Q_1}$ is a family of K-linear maps $K^d \to K^d$, each taking K_l to $\bigoplus_{u \geq l+1} K_u$. (Here $K_{L+1} = 0$.) Then $(f_\alpha)_{\alpha \in Q_1}$ is a point in $\mathbf{Rep}_d(\Lambda)$, and the corresponding Λ-module M has radical layering $\mathbb{S}(M) \geq \mathbb{S}$.

Moreover, for any $m \in \{0, \dots, L\}$, the following conditions are equivalent:

- *The first m entries of $\mathbb{S}(M)$ are $\mathbb{S}_0, \dots, \mathbb{S}_m$ in this order.*
- *For $l \in \{0, \dots, m\}$, the subspace $\sum_{q \in Q_{>0}} f_q(K_l)$ contains $\bigoplus_{u \geq l+1} K_u$.*

In particular, $\mathbb{S}(M) = \mathbb{S}$ if and only if the following holds for $0 \leq l < L$: Modulo $\bigoplus_{u \geq l+2} K_u$, the sum $\sum_{\alpha \in Q_1} f_\alpha(K_l)$ has the same dimension as $\mathbb{S}_{l+1}$.

Proof. The first assertion is immediate from the fact that $f_p(K^d) = 0$ whenever p is a path of length $\geq L+1$. To see that the Λ-module M corresponding to the family $(f_\alpha)_{\alpha \in Q_1}$ of linear maps satisfies $\mathbb{S}(M) \geq \mathbb{S}$, observe that, for any $l \in \{1, \dots, L\}$, the K-vector space $J^l M$ equals $\sum f_p(K^d)$, where the sum extends over all paths p of length $\geq l$; by hypothesis, this sum is contained in $\bigoplus_{j \geq l} K_j$. Verification of the equivalences is straightforward. $\square$

The next lemma provides us with a means to exclude certain types of semisimple sequences from the set of potential generic radical layerings of irreducible components of $\mathbf{Rep}_d(\Lambda)$.

Lemma 5.2. *Let $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ be a realizable semisimple sequence. Suppose there exists an index $\rho \in \{0, \dots, L-1\}$ such that, for every Λ-module M in $\mathbf{Rep}\,\mathbb{S}$, the subfactor $J^\rho M / J^{\rho+2} M$ has a simple direct summand. Moreover, let $\mathbb{S}_\rho = \widetilde{\mathbb{S}}_\rho \oplus S$ and $\widetilde{\mathbb{S}}_{\rho+1} = \mathbb{S}_{\rho+1} \oplus S$, and suppose that the sequence*

$$\widetilde{\mathbb{S}} = (\mathbb{S}_0, \dots, \mathbb{S}_{\rho-1}, \widetilde{\mathbb{S}}_\rho, \widetilde{\mathbb{S}}_{\rho+1}, \mathbb{S}_{\rho+2}, \dots, \mathbb{S}_L)$$

is again realizable. Then $\mathbf{Rep}\,\mathbb{S}$ is contained in $\overline{\mathbf{Rep}\,\widetilde{\mathbb{S}}}$.

Proof. Let M be any left Λ-module with radical layering $\mathbb{S}$. Choose a basis

$$B = (b_{l,j})_{0 \leq l \leq L,\, 1 \leq j \leq \dim \mathbb{S}_l}$$

for M, with the property that, for $0 \leq l \leq L$, the elements $b_{l,1}, \dots, b_{l, \dim \mathbb{S}_l}$ form a basis for $J^l M$ modulo $J^{l+1} M$. The first hypothesis allows us to assume that $J\, b_{\rho,1}$ is contained in $J^{\rho+2} M$.

In light of Criterion 3.2, our realizability hypothesis guarantees that the semisimple sequence $(\widetilde{\mathbb{S}}_\rho, \widetilde{\mathbb{S}}_{\rho+1}) = (S^{\dim \mathbb{S}_\rho - 1}, \mathbb{S}_{\rho+1} \oplus S)$ is again realizable and that $\dim \mathbb{S}_{\rho+1} + 1 \leq r \cdot (\dim \mathbb{S}_\rho - 1)$. Since $J^{\rho+1} M / J^{\rho+2} M = \mathbb{S}_{\rho+1}$ and $J b_{\rho,1} = 0$ modulo $J^{\rho+2} M$, this amounts to the existence of an index $s \geq 2$ and an arrow $\gamma \in Q_1$ such that

$$(\dagger) \qquad \gamma\, b_{\rho,s} \quad \in \sum_{\substack{\alpha \in Q_1,\, j \geq 2 \\ \alpha \neq \gamma \text{ or } j \neq s}} K\, \alpha\, b_{\rho,j} \quad \text{modulo } J^{\rho+2} M.$$

In particular, $\dim \mathbb{S}_\rho \geq 2$. It is clearly innocuous to assume $s = 2$.

Using Lemma 5.1, we will construct a deformation $(D_t)_{t \in K}$ of M in Λ-mod such that $D_0 \cong M$, while $\mathbb{S}(D_t) = \widetilde{\mathbb{S}}$ for all $t \neq 0$. Our construction will amount to specifying a morphism $\mathbb{A}^1 \to \mathbf{Rep}_d(\Lambda)$ which sends any $t \in K$ to a point $(f_\alpha^{(t)})_{\alpha \in Q_1}$ in $\mathrm{Rep}\,\widetilde{\mathbb{S}}$. This will place M into the closure of $\mathrm{Rep}\,\widetilde{\mathbb{S}}$ in $\mathbf{Rep}_d(\Lambda)$, whence we will obtain the desired containment $\mathbf{Rep}\,\mathbb{S} \subseteq \overline{\mathbf{Rep}\,\widetilde{\mathbb{S}}}$.

We first pin down basis expansions of the products $\alpha\, b_{l,j}$ in M, where α traces the arrows of Q, namely

$$\alpha\, b_{l,j} \;=\; \sum_{u \geq l+1,\, v \leq \dim \mathbb{S}_u} c_{u,v}^{(\alpha,l,j)}\; b_{u,v}$$

with scalars $c_{u,v}^{(\alpha,l,j)}$. For an unambiguous introduction of the K-linear maps $f_\alpha^{(t)}$, we rename the basis B for K^d to $\widetilde{B} = (\widetilde{b}_{l,j})$. In the spirit of Lemma 5.1 (relative to $\widetilde{\mathbb{S}}$), we introduce the vector space decomposition

$$K^d \;=\; \bigoplus_{0\le l\le L} K_l,$$

where the K_l are as follows: If $l \notin \{\rho, \rho+1\}$, we let K_l be the subspace generated by the $\widetilde{b}_{l,j}$, $1 \le j \le \dim \mathbb{S}_l$. Moreover, K_ρ is defined as the span of $\widetilde{b}_{\rho,2}, \dots, \widetilde{b}_{\rho,\dim \mathbb{S}_\rho}$, and $K_{\rho+1}$ as the span of the $(\dim \mathbb{S}_{\rho+1} + 1)$ elements $\widetilde{b}_{\rho,1}$ and $\widetilde{b}_{\rho+1,j}$ for $1 \le j \le \dim \mathbb{S}_{\rho+1}$.

Our choice of $b_{\rho,1}$ entails

$$(\dagger\dagger) \qquad c_{\rho+1,v}^{(\alpha,\rho,1)} = 0 \text{ for all arrows } \alpha \text{ and all } v \le \dim \mathbb{S}_{\rho+1}.$$

These equalities guarantee that the following K-endomorphisms $f_\alpha^{(t)}$ of K^d satisfy the hypotheses of Lemma 5.1. Namely, if α is an arrow in $Q_1 \setminus \{\gamma\}$, we define

$$f_\alpha^{(t)}(\widetilde{b}_{l,j}) \;=\; \sum_{u \ge l+1,\, v \le \dim \mathbb{S}_u} c_{u,v}^{(\alpha,l,j)}\; \widetilde{b}_{u,v}.$$

Moreover, we set $f_\gamma^{(t)}(\widetilde{b}_{l,j}) = \sum_{u\ge l+1,\, v\le \dim \mathbb{S}_u} c_{u,v}^{(\gamma,l,j)}\; \widetilde{b}_{u,v}$ if either $l \ne \rho$ or $j \ne 2$, and supplement by

$$f_\gamma^{(t)}(\widetilde{b}_{\rho,2}) \;=\; t\cdot \widetilde{b}_{\rho,1} \;+\; \sum_{u\ge \rho+1,\, v\le \dim \mathbb{S}_u} c_{u,\,v}^{(\gamma,\rho,2)}\; \widetilde{b}_{u,v}.$$

Note that $f_\alpha^{(t)}(\widetilde{b}_{\rho,1}) \in \bigoplus_{l\ge \rho+2} K_l$ by $(\dagger\dagger)$, which shows that, for any arrow $\alpha \in Q_1$, we have $f_\alpha^{(t)}(K_{\rho+1}) \subseteq \bigoplus_{l\ge\rho+2} K_l$. This makes the family of maps comply with the setup of Lemma 5.1. Let D_t be the Λ-module defined by the maps $f_\alpha^{(t)}$. That $D_0 \cong M$ is clear. We now apply one implication of the final statement of Lemma 5.1 to the K-linear maps $K^d \to K^d$, $x \mapsto \alpha x$ defining M, and the other implication to the maps $f_\alpha^{(t)}$ defining D_t, in order to conclude that D_t has radical layering $\widetilde{\mathbb{S}}$ whenever $t \ne 0$. Indeed, from $(\dagger)$ and the construction of the $f_\alpha^{(t)}$ we deduce that

$$\sum_{u\ge\rho+1,\, v\le \dim \mathbb{S}_u} c_{u,\,v}^{(\gamma,\rho,2)}\; \widetilde{b}_{u,v} \;\in\; \sum_{\substack{\alpha\in Q_1,\, j\ge 2\\ \alpha\ne\gamma \text{ or } j\ne 2}} K\, f_\alpha^{(t)}(\widetilde{b}_{\rho,j}) \quad \text{modulo} \bigoplus_{l\ge\rho+2} K_l\,,$$

whence $\widetilde{b}_{\rho,1}$ belongs to

$$\sum_{j\ge 2} K f_\gamma^{(t)}(\widetilde{b}_{\rho,j}) \;+\; \sum_{\alpha\in Q_1,\, \alpha\ne\gamma,\, j\ge 2} K f_\alpha^{(t)}(\widetilde{b}_{\rho,j}) \quad \text{modulo} \bigoplus_{l\ge\rho+2} K_l$$

for $t \neq 0$. Therefore $K_{\rho+1}$ is contained in $\sum_{q \in Q_{>0}} f_q^{(t)}(K_\rho)$. We conclude that, for $t \neq 0$ and $l \geq 1$, the K-space $\sum_{\alpha \in Q_1} f_\alpha^{(t)}(K_{l-1})$ equals K_l, modulo $\bigoplus_{u \geq l+1} K_u$; but $\dim K_l = \dim \widetilde{\mathbb{S}}_l$ by definition. As we pointed out earlier, this proves our claim. $\square$

We remark that Lemma 5.2 does not apply to any $\rho \in \{0, \dots, L-1\}$ with the property that $\mathbb{S}_\rho$ is simple, since the corresponding semisimple sequence $\widetilde{\mathbb{S}}$ fails to be realizable in this situation. Hence the case of low dimensions d will require separate consideration in the proof of the Main Theorem.

Proof of the Main Theorem. We subdivide the argument into several steps.

Step 1. In this preliminary step, we let d be any positive integer and suppose $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ to be realizable with $\dim \mathbb{S} = d$. We denote the closure of $\mathbf{Rep}\,\mathbb{S}$ in $\mathbf{Rep}_d(\Lambda)$ by $\mathfrak{U}$ and set $\mathfrak{m} = \min(d, L+1)$. Our goal is to show that the irreducible variety $\mathfrak{U}$ fails to be an irreducible component of $\mathbf{Rep}_d(\Lambda)$ unless $\mathbb{S}_{\mathfrak{m}-1} \neq 0$.

Assume $\mathbb{S}_{\mathfrak{m}-1} = 0$, let L' be maximal with respect to $\mathbb{S}_{L'} \neq 0$, and pick $h \leq L'$ maximal with respect to the condition that $\mathbb{S}_h$ is not simple; due to our assumption, such an index h exists. Given a module M in $\mathbf{Rep}\,\mathbb{S}$, we choose a basis $B = (b_{l,j})$ for M, where $l \in \{0, \dots L'\}$ and $j \in \{1, \dots, \dim \mathbb{S}_l\}$, such that the $b_{l,j}$ form a basis for $J^l M$ modulo $J^{l+1}M$. For all arrows α and all legitimate choices of l, j, we then obtain expansions

$$(\dagger) \qquad \alpha\, b_{l,j} = \sum_{l+1 \leq u \leq L',\, v \leq \dim \mathbb{S}_u} c_{u,v}^{(\alpha,l,j)}\ b_{u,v}$$

with $c_{u,v}^{(\alpha,l,j)} \in K$. Since $\dim \mathbb{S}_h \geq 2$, while $\dim \mathbb{S}_{h+1} \in \{0, 1\}$ by our choice of h, we may assume – on possibly adjusting the basis elements $b_{h,j}$ for $J^h M$ modulo $J^{h+1}M$ – that $\alpha_1 b_{h,1} \in J^{h+2}M$; indeed, since $\dim \mathbb{S}_{h+1} = \dim J^{h+1}M/J^{h+2}M < \dim \mathbb{S}_h$, the elements $\alpha_1 b_{h,j}$ with $1 \leq j \leq \dim \mathbb{S}_h$ are linearly dependent modulo $J^{h+2}M$.

Next we ascertain the existence of an index $s \geq 2$ with the property that $J^{h+1}M = Jb_{h,s}$. From Theorem 3.6 we glean that, generically, the modules N in $\mathbf{Rep}\,\mathbb{S}$ (and hence the modules in $\mathfrak{U}$) satisfy the following condition for each $l \leq L$ with $\mathbb{S}_{l+1} \neq 0$: For any arrow $\alpha \in Q_1$, there exists an element $x \in J^l N \setminus J^{l+1}N$ with the property that $\alpha\, x \in J^{l+1}N \setminus J^{l+2}N$. In showing that $\mathbf{Rep}\,\mathbb{S}$ is properly contained in some closure $\overline{\mathbf{Rep}\,\widetilde{\mathbb{S}}}$ for a sequence $\widetilde{\mathbb{S}} < \mathbb{S}$ to be specified, it consequently suffices to verify that the modules $N \in \mathbf{Rep}\,\mathbb{S}$ with this latter property belong to the variety $\overline{\mathbf{Rep}\,\widetilde{\mathbb{S}}}$. We return to the task at hand: If $J^{h+1}M = 0$, there is nothing to be shown. So suppose otherwise. $J^{h+1}M$ then being a nontrivial uniserial module, it suffices to ensure $Jb_{h,s} \not\subseteq J^{h+2}M$ for some $s \geq 2$. But in light of $\alpha_1 b_{h,1} \in J^{h+2}M$, the preceding considerations legitimize the assumption that $\alpha_1 \widetilde{b}_{h,s} \in J^{h+1}M \setminus J^{h+2}M$ for some $s \geq 2$. It is clearly harmless to settle on $s = 2$. In the sequel, we may thus assume

$$(\dagger\dagger) \qquad J^{h+1}M = Jb_{h,2}.$$

Now we construct a family $(D_t)_{t \in K}$ of d-dimensional Λ-modules such that $D_0 \cong M$ and D_t has the following radical layering $\widetilde{\mathbb{S}}$ for $t \neq 0$: Namely, $\widetilde{\mathbb{S}}_l = \mathbb{S}_l$ for $l < h$, $\widetilde{\mathbb{S}}_h = S^{\dim \mathbb{S}_h - 1}$,

$\widetilde{\mathbb{S}}_l = S$ for $h+1 \le l \le L'+1$, and $\widetilde{\mathbb{S}}_l = 0$ for $l > L'+1$. Note that, by our assumption, $L'+1 \le L$.

The sequence $\mathbb{S}$ being realizable, Criterion 3.2 shows the same to be true for $\widetilde{\mathbb{S}}$. To unambiguously introduce the D_t, we rename the basis B to $\widetilde{B} = (\widetilde{b}_{l,j})$, cautioning, however, that the indexing of $\widetilde{B}$ will not be in alignment with the radical layering $\widetilde{\mathbb{S}}$ of D_t for $t \neq 0$; the element $b_{h,2}$ in the h-th radical layer of M will be moved downward to turn into the reincarnation $\widetilde{b}_{h,2}$ in the $(h+1)$-st layer of D_t, and this will typically trigger further downward shifts. In order to display a point $\big(f_{\alpha_i}\big)_{i\le r} \in \mathbf{Rep}_d(\Lambda)$ that gives rise to D_t by way of Lemma 5.1, we consider the following direct sum decomposition of d-dimensional K-space: $K^d = \bigoplus_{0\le l\le L'} K_l$, where $K_l = \bigoplus_{j\le \dim \mathbb{S}_l} K\widetilde{b}_{l,j}$ for $l < h$, $K_h = \bigoplus_{j\le \dim \mathbb{S}_h,\, j\neq 2} K\widetilde{b}_{h,j}$, $K_{h+1} = K\widetilde{b}_{h,2}$, and $K_l = K\widetilde{b}_{l-1,1}$ for $h+2 \le l \le L'+1$. We start with the interloper assignment:

$$(\dagger\,\dagger\,\dagger) \qquad f^{(t)}_{\alpha_1}(\widetilde{b}_{h,1}) \;=\; t\cdot \widetilde{b}_{h,2} \;+\; \sum_{h+2\le u\le L'} c^{(\alpha_1,h,1)}_{u,1}\, \widetilde{b}_{u,1}.$$

To justify the range of the summation on the right of $(\dagger\dagger\dagger)$, keep in mind that $c^{(\alpha_1,h,1)}_{h+1,v} = 0$ for $v \le \dim \mathbb{S}_{h+1}$ by construction, and $\dim \mathbb{S}_u = 1$ for $h+2 \le u \le L'$. Beyond $(\dagger\,\dagger\,\dagger)$, the linear maps $K_l \to \bigoplus_{j\ge l+1} K_j$ formally duplicate those giving rise to the Λ-module structure M on K^d; see $(\dagger)$. Explicitly, the $f_\alpha : K^d \to K^d$ are defined via

$$f^{(t)}_{\alpha}(\widetilde{b}_{l,j}) \;=\; \sum_{u\ge l+1,\, v\le \dim \mathbb{S}_u} c^{(\alpha,l,j)}_{u,v}\, \widetilde{b}_{u,v} \quad \text{for} \quad (\alpha,l,j) \neq (\alpha_1,h,1).$$

It is readily checked that the r-tuple $(f^{(t)}_{\alpha})_{\alpha\in Q_1}$ satisfies the conditions of Lemma 5.1, so as to yield a Λ-module D_t with $\mathbb{S}(D_t) = \widetilde{\mathbb{S}}$ for $t \neq 0$. Indeed, to verify this last equality, note that, for $t \neq 0$, equation $(\dagger\,\dagger\,\dagger)$ places $\widetilde{b}_{h,2}$ into $J^{h+1}D_t \setminus J^{h+2}D_t$; then use $(\dagger\dagger)$ to show that the layers $\mathbb{S}_l(D_t)$ for $l \ge h+2$ are as required. It is obvious that $D_0 \cong M$.

Clearly, the map $\mathbb{A}^1 \to \mathbf{Rep}_d(\Lambda)$ which sends t to the point $\big(f^{(t)}_{\alpha_i}\big) \in \mathbf{Rep}_d(\Lambda)$ specified above is a morphism of varieties. This ensures that M belongs to the closure of $\operatorname{Rep}\widetilde{\mathbb{S}}$ in $\mathbf{Rep}_d(\Lambda)$, in turn an irreducible subvariety of $\mathbf{Rep}_d(\Lambda)$. Given the $\mathfrak{U}$-generic choice of M, we conclude that $\mathfrak{U} \subsetneqq \overline{\mathbf{Rep}\,\widetilde{\mathbb{S}}}$. Thus $\mathfrak{U}$ fails to be an an irreducible component of $\mathbf{Rep}_d(\Lambda)$ as claimed.

Step 2. In this step, we prove assertion (II) of the Main Theorem. So let $d \le L+1$. By Step 1, only the semisimple sequence $\mathbb{S}$ with $\mathbb{S}_l = S$ for $0 \le l \le d-1$ and $\mathbb{S}_l = 0$ for $l \ge d$ has the potential of being a generic radical layering of the modules in an irreducible component of $\mathbf{Rep}_d(\Lambda)$. Consequently, $\mathbb{S}$ is the unique generic one for $\mathbf{Rep}_d(\Lambda)$, meaning that $\mathbf{Rep}_{\mathbf{d}}(\Lambda) = \overline{\mathbf{Rep}\,\mathbb{S}}$ is irreducible and the modules in $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$ are generically uniserial. In particular, the generic socle layering $\mathbb{S}^*$ of $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$ equals $\mathbb{S}$, making the pair $(\mathbb{S},\mathbb{S}^*)$ the unique smallest element of $\mathbf{rad\text{-}soc(d)}$. This justifies part (II) of the Main Theorem.

Step 3. Finally, we verify the equivalence of conditions (1) – (4) under the hypothesis that $d > L + 1$. The equivalence "(1) $\iff$ (1')" follows from Proposition 2.1. Next we recall that realizability of $\mathbb{S}$ is tantamount to $\dim \mathbb{S}_l \le r \cdot \dim \mathbb{S}_{l-1}$ for $1 \le l \le L$.

"(2) $\Longrightarrow$ (3)". Assume that (2) holds. Then $\mathbb{S}$ is realizable by the first set of inequalities. Let E and E_l be injective envelopes of S and $\mathbb{S}_l$, respectively. Since $\mathrm{soc}_1(E)/S \cong S^r$, the companion inequalities, $r \cdot \dim \mathbb{S}_l \ge \dim \mathbb{S}_{l-1}$ for $l \ge 1$, guarantee that the semisimple module $\mathbb{S}_{l-1}$ embeds into the first socle layer $\mathbb{S}_1^*(E_l) = \mathrm{soc}_1(E_l)/\mathrm{soc}_0(E_l)$. Using Lemma 3.8, we conclude that $(\mathbb{S}_L, \dots, \mathbb{S}_0)$ is indeed the generic socle layering of the variety $\mathbf{Rep}\,\mathbb{S}$.

The implication "(3) $\Longrightarrow$ (4)" follows from Lemma 2.4(a), and "(4) $\Longrightarrow$ (1)" is covered by Theorem 3.1.

To verify "(1) $\Longrightarrow$ (2)", we assume (2) to fail. If $\dim \mathbb{S}_l > r \cdot \dim \mathbb{S}_{l-1}$ for some $l \in \{1, \dots, L\}$, then $\mathbf{Rep}\,\mathbb{S}$ is empty as remarked at the outset, and (1) fails as well. So, in showing that the irreducible subvariety $\mathcal{C} = \overline{\mathbf{Rep}\,\mathbb{S}}$ of $\mathbf{Rep}_d(\Lambda)$ is not an irreducible component, we may assume $\dim S_l \le r \cdot \dim \mathbb{S}_{l-1}$ for all eligible l. Our assumption then yields an index $\rho \in \{0, \dots, L-1\}$ with the property that $\dim \mathbb{S}_\rho > r \cdot \dim \mathbb{S}_{\rho+1}$ (†). From Step 1, we moreover know that the equality $\mathbb{S}_L = 0$ excludes the possibility that $\mathcal{C} = \overline{\mathbf{Rep}\,\mathbb{S}}$ be an irreducible component of $\mathbf{Rep}_d(\Lambda)$. In ascertaining that $\mathcal{C}$ indeed fails to be such a component, it is therefore harmless to additionally assume $\mathbb{S}_L \neq 0$.

From (†) we infer that $\mathbb{S}_\rho$ does not embed into $\mathrm{soc}_1(E_{\rho+1})/\mathrm{soc}_0(E_{\rho+1})$, where $E_{\rho+1}$ is an injective envelope of $\mathbb{S}_{\rho+1}$. Consequently, we obtain: • Every Λ-module with radical layering $(\mathbb{S}_\rho, \mathbb{S}_{\rho+1})$ has a simple direct summand, and • the sequence

$$\widetilde{\mathbb{S}} \;=\; (\mathbb{S}_0, \dots, \mathbb{S}_{\rho-1}, S^{\dim \mathbb{S}_\rho - 1}, \mathbb{S}_{\rho+1} \oplus S, \mathbb{S}_{\rho+2}, \dots, \mathbb{S}_L)$$

is again realizable; keep in mind that $r \ge 2$. Hence Lemma 5.2 implies that $\overline{\mathbf{Rep}\,\mathbb{S}}$ is (properly) contained in $\overline{\mathbf{Rep}\,\widetilde{\mathbb{S}}}$. The latter variety being in turn irreducible, this rules out the possibility that $\overline{\mathbf{Rep}\,\mathbb{S}}$ is an irreducible component of $\mathbf{Rep}_d(\Lambda)$, and the argument is complete. □

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