Elementary Partial Differential Equations and Applications

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Contents

List of Figures

Introduction

This book grew out of a two-quarter sequence of undergraduate courses offered at the University of California (UCSB), for science majors, engineers and mathematicians. These courses along with a two-quarter sequence on ordinary differential equations (ODEs) and dynamical systems constitute the applied mathematics courses for the *Program in Scientific Computations*, a joint program between the mathematics department and the College of Engineering at UCSB. More information about this program can be obtained at the URL address: www.math.ucsb.edu/~birnir.

One could call the approach taken in this book a qualitative theory of linear PDEs and that is what the more theoretically inclined students can gain from it. A similar approach to nonlinear PDEs is the basis for a rapidly emerging qualitiative theory of nonlinear PDEs that is currently having a great impact on many fields of modern science. Fourier series (seperation of variables) are absent in this volume and this was done on purpose. The philosophy is that Fourier series give different insight into boundary value problems and should be taught seperately. Fourier series and numerical methods of PDEs are also treated in the courses mentioned above, but appear in the sequel to this volume.

This book is designed for students who will eventually be solving partial differential equations (PDEs) numerically. The aim is to teach them enough appreciation for the properties of the solutions to be able to judge with confidence, whether their numerical solutions make sense and how to interpret them. The examples, applications and exercises included in this book should give students a good intuition for the distinct qualities of waves, heat diffusion and fields.

The point of view presented in the book was not developed by the author. It originated with David Hilbert in Göttingen, at the turn of the last century, and the author learned it from his fellow graduate students at the Courant Institute in New York, in the late seventies. They introduced him to a wonderful book: Techniques in Partial Differential Equations, by C. R. Chester [2], that has unfortunately been out of print for many years. The other book the author owes much to is: Fourier Series and Integrals, by H. P. McKean and H. Dym, [5]. The contribution that the author wants to make to the material presented in these books, is to show how simple and elegant linear PDE theory is, if one just approaches it in the right way.

The author is also indepted to Carolyn Johnson for her excellent typesetting and his students at the University of Iceland and UCSB who taught him how to explain it all.

Café Roma, Isla Vista, September 13, 1998.

Chapter 1

The Wave Equation

1.1 The Wave, the Heat, the Potential and the Schrödinger equations

Four partial differential equations play a special role in the sciences. They are,

the Wave Equation

$$u_{tt} = c^2 \Delta u,$$

the Heat Equation

$$u_t = k \Delta u,$$

the Potential Equation

$$\Delta u = 0,$$

and Schrödinger's Equation

$$i\hbar \psi_t = -\frac{\hbar^2}{2m} \Delta \psi.$$

$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is called the Laplace operator. The wave equation describes the propagation of waves in some medium or field, c is the speed of propagation. The heat equation describes the time evolution of the heat distribution in materials, k is the conductivity. The potential equation describes, for example, the displacement of the surface of a drum, and electric and fluid potentials. The solution, $\psi(x,t)$ of Schrödinger's Equation is the wave function of a free particle, such as an electron with mass m, moving through space, $\hbar$ is Planck's constant. ψ will determine the expected value of any observable, such as position or momentum of the particle, at time t; for example

$$p(t) = \langle \psi | P\psi \rangle = \int_{-\infty}^{\infty} \psi i\hbar \overline{\frac{\partial \psi}{\partial x}}\, dx,$$

where $P = i\hbar\partial/\partial x$ is called the momentum operator.

We can describe various physical phenomena that we are interested in, if we can solve partial differential equations. For example, one can sometimes write an electric field as the gradient of an electric potential

$$E = -\nabla V.$$

Ohm's law then says that the current is

$$I = \sigma E,$$

where σ is the conductivity and if no charge is present, then the divergence of I vanishes

$$\nabla \cdot I = 0.$$

Now

$$\nabla \cdot I = \sigma \nabla \cdot E = -\sigma \nabla \cdot \nabla V = -\sigma \Delta V$$

so

$$\Delta V = 0,$$

or the electric potential satisfies the potential equation. This means that if we solve the potential equation for V then we can find the electric field E and the current I.

1.2 The Derivation of the Wave Equation

Consider a violin string with length ℓ, where the (vertical) distance $u(x,t)$ from the violin is a function of time t and the distance x from the end of the string, $0 \leq x \leq \ell$, see Figure 1.1. We pick a point (x,u)

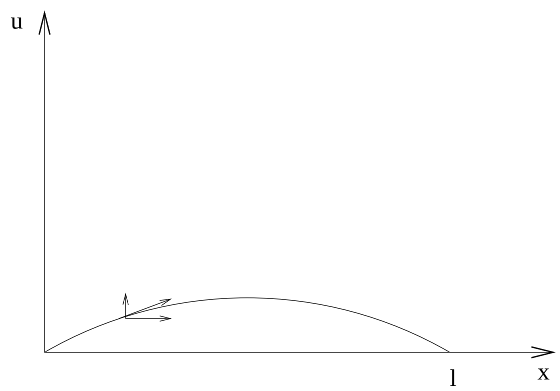

Figure 1.1: The violin string.

on the string and apply Newton's law

$$\text{Mass} \times \text{Acceleration} = \text{Force}$$

or

$$\rho(x)(b-a)u_{tt} = f(x).$$

We take a piece of the string of length $b-a$, then if $\rho(x)$ is the density of the string, measured in mass per unit length, the mass of this piece of string is $\rho(b-a)$. We assume that the tension $T(x)$ in the string is directed along the tangent to the string. This means that the string is flexible and does not resist being bent. Moreover, it is too light for the gravitational force acting on it to be significant. Then the vertical acceleration balances the vertical part of the tension, see Figure 1.2,

$$T_u = \frac{Tu_x}{\sqrt{1+u_x^2}}.$$

This is simply saying that the unit tangent vector to the string at the point (x,u), see Figure 1.2, is

$$\left(\frac{1}{\sqrt{1+u_x^2}}, \frac{u_x}{\sqrt{1+u_x^2}}\right).$$

We assume that the string is homogeneous so that both the tension $T(x) = T$ and the density $\rho(x) = \rho$ are independent of the position along the string. Then we get the net force acting on a piece of string extending from $x = a$ to $x = b$,

$$f(x) = T_u|_{x=a}^{x=b} = \frac{Tu_x}{\sqrt{1+u_x^2}}|_{x=a}^{x=b}.$$

Now Newton's law gives us the equation

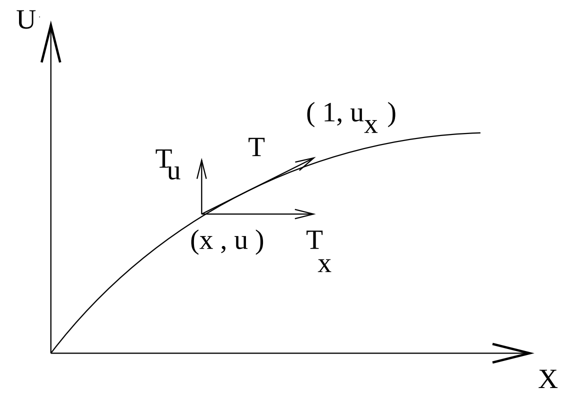

Figure 1.2: The tension in the violin string.

$$\rho u_{tt} = \frac{T}{(b-a)}\left[\frac{u_x(b,t)}{\sqrt{1+u_x^2(b,t)}} - \frac{u_x(a,t)}{\sqrt{1+u_x^2(a,t)}}\right],$$

and if we shorten the piece of string by making $b \to a$, we get

$$\rho u_{tt} = T\frac{\partial}{\partial x}\left[\frac{u_x}{\sqrt{1+u_x^2}}\right],$$

because the difference quotient becomes the derivative as $b-a = \Delta x \to 0$. If in addition the amplitude of the string oscillations is small, then u_x^2 is much smaller than u_x and $\sqrt{1+u_x^2} \approx 1$. This approximation gives us the wave equation

$$u_{tt} = \frac{T}{\rho}u_{xx} = c^2 u_{xx}.$$

The wave speed squared $c^2 = \frac{T}{\rho} > 0$ is a positive quantity because both the tension T and the density ρ are positive quantities. The velocity c has the dimensions of distance over time, because

$$T = \frac{md}{t^2}, \quad \rho = \frac{m}{d},$$

where d denotes distance and

$$c^2 = \frac{T}{\rho} = \frac{md}{\frac{mt^2}{d}} = \frac{d^2}{t^2}.$$

1.3 The d'Alembert Solution of the Wave Equation

It is natural to specify the initial displacement of a violin string

$$u(x,0) = f(x),$$

if we want to find the displacement of the string as a function of time and position along the string. In addition we must specify the initial velocity

$$u_t(x,0) = g(x).$$

This gives us an initial value problem (IVP) for the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R},\ t \in \mathbb{R}$$

with the initial conditions (IC),

$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

We have let the string extend all the way from $-\infty$ to ∞. Initial value problems are natural for hyperbolic PDEs, such as the wave equation, and parabolic equations, such as the heat equation. We will see later that boundary value problems (BVP) are more appropriate for elliptic equations such as the potential equation. In chapter 2, we will discover where the terminology, hyperbolic, parabolic and elliptic PDE originates.

The initial value problem for the wave equation is solved by use of characteristic coordinates

$$\xi = x - ct, \quad \eta = x + ct.$$

We apply the chain rule to find the derivatives of u with respect to ξ and η,

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta, \\ u_t &= u_\xi \xi_t + u_\eta \eta_t = c(-u_\xi + u_\eta). \end{aligned}$$

Another way of saying this is in terms of the operators,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial t} = c\left(-\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right),$$

and the second derivative operators become,

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= \left(\frac{\partial}{\partial \xi} + \frac{\partial}{2\eta}\right)^2 = \frac{\partial^2}{\partial \xi^2} + \frac{2\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}, \\ \frac{\partial^2}{\partial t^2} &= c^2\left(-\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)^2 = c^2\left[\frac{\partial^2}{\partial \xi^2} - \frac{2\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}\right]. \end{aligned}$$

Consequently,

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= c^2[u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} - u_{\xi\xi} - 2u_{\xi\eta} - u_{\eta\eta}] \\ &= -4c^2 u_{\xi\eta}. \end{aligned}$$

We see that in characteristic coordinates the wave equation takes the simple form

$$u_{\xi\eta} = 0.$$

This equation is easily solved by integration. First we integrate with respect to η,

$$u_\xi = \text{a constant function of } \eta = \phi'(\xi).$$

Then we integrate with respect to ξ,

$$\begin{aligned} u &= \phi(\xi) + \text{a constant function of } \xi, \\ &= \phi(\xi) + \psi(\eta). \end{aligned}$$

Thus we see that in characteristic coordinates the solution of the wave equation can be written

$$u(\xi, \eta) = \phi(\xi) + \psi(\eta),$$

and in the original coordinates as

$$u(x, t) = \phi(x - ct) + \psi(x + ct).$$

This solution is the sum of two waves, $\phi(x - ct)$ that moves to the right and $\psi(x + ct)$ that moves to the left. ϕ and ψ are arbitrary functions so far, below they will be determined by the initial conditions.

At this state we should figure out where the term characteristic coordinate originates. $x - ct =$ constant, determines a line with positive slope, $c > 0$, in the $x - t$ plane, see Figure 1.3. ϕ is constant on this line. Similarly ψ is constant on the line $x + ct =$ constant, which has a negative slope in the $x - t$ plane, see Figure 1.3. The two lines,

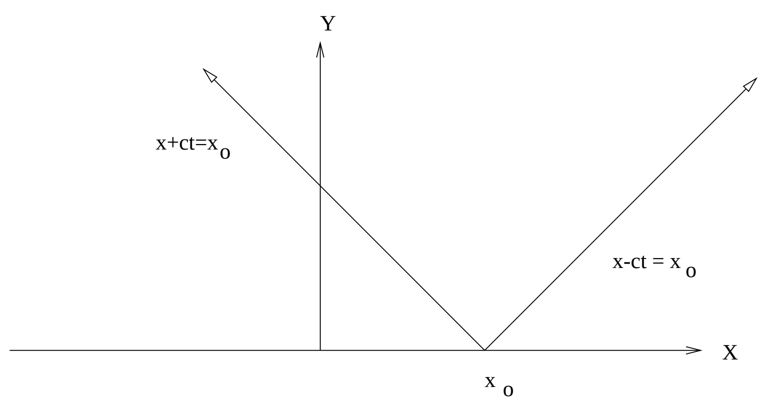

Figure 1.3: The two characteristics.

$x \pm ct =$ constant, are called the characteristics, because they carry the initial values from their origin on the initial line $t = 0$. In fact this property, that initial data is carried along characteristics, distinguishes hyperbolic PDEs from their parabolic and elliptic counterparts.

Now we use the initial conditions in the IVP to find ϕ and ψ. A substitution into the IC gives

$$\begin{aligned} u(x, 0) &= \phi(x) + \psi(x) = f(x) \\ u_t(x, 0) &= -c\phi'(x) + c\psi'(x) = g(x). \end{aligned}$$

We differentiate the first equation and solve for ϕ' and ψ',

$$\begin{aligned}\phi'(x) &= \frac{1}{2}f'(x) - \frac{1}{2c}g(x),\\ \psi'(x) &= \frac{1}{2}f'(x) + \frac{1}{2c}g(x).\end{aligned}$$

Then we integrate these two equations to get,

$$\begin{aligned}\phi(x) &= \frac{f(x)}{2} - \frac{1}{2c}\int_0^x g(x)\,ds + k_1,\\ \psi(x) &= \frac{f(x)}{2} + \frac{1}{2c}\int_0^x g(x)\,ds + k_2,\end{aligned}$$

where k_1 and k_2 are constants. This implies that

$$\phi(x-ct)+\psi(x+ct) = \frac{f(x-ct)+f(x+ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds+k_1+k_2,$$

and the IC

$$\phi(x)+\psi(x) = f(x),$$

shows that $k_1+k_2=0$. Thus we get the d'Alembert solution of the wave equation

$$u(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds. \qquad (1.1)$$

Exercises 1.3.1

1.1 Show that $\phi(x-ct)$ is a wave moving to the right, $\psi(x+ct)$ a wave moving to the left.

Hint: Let ϕ and ψ be a small bump, see Figure 1.4, centered at the origin, when $t=0$.

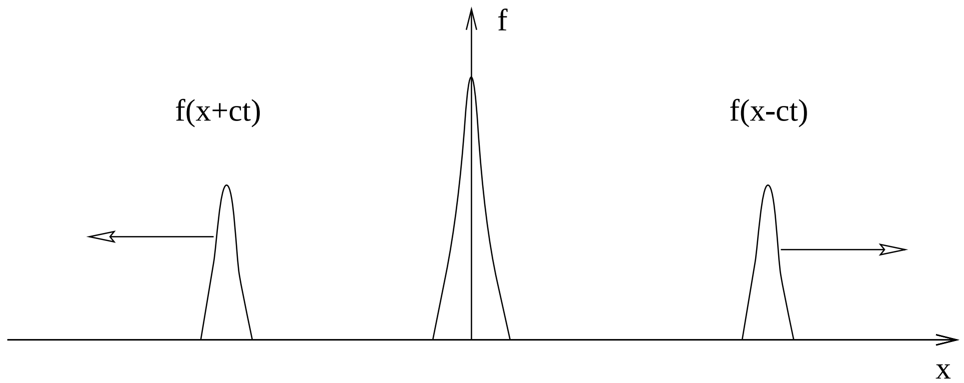

Figure 1.4: The left- and right-travelling waves.

1.4 Properties of One-dimensional Waves

We can study the properties of the d'Alembert solution by considering a point(x_0, t_0) in the $x - t$ plane. If we draw the lines

$$x - c\,t = x_0 - c\,t_0 \quad \text{and} \quad x + c\,t = x_0 + c\,t_0$$

through the point (x_0, t_0), see Figure 1.5, then the ϕ is constant on the former line and the ψ is constant on the latter line. One can say that the solution is carried along these two characteristics from the

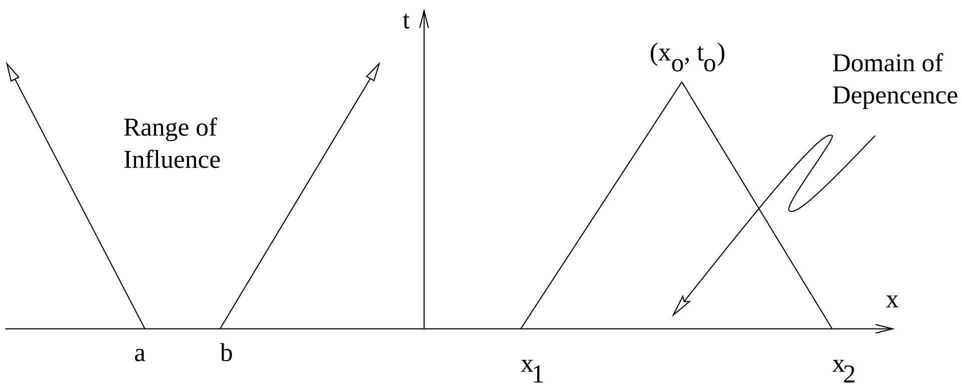

Figure 1.5: The domain of dependence and the range of influence.

intersection points

$$x_1 = x_0 - ct_0 \quad \text{and} \quad x_2 = x_0 + ct_0$$

of the lines with the x axis, see Figure 1.5. If we look at the form of the solution

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds,$$

then we see that it is obtained by taking the average of the amplitude f at these two points and adding to it the integral of the velocity g over the interval $[x_1, x_2]$ between the two intersection points. Thus the solution in the point (x_0, t_0) is determined by the initial conditions (data) on the interval $[x_1, x_2]$. This interval is called *the domain of dependence* of (x_0, t_0), see Figure 1.5. We also see, on Figure 1.5, that the initial conditions on the interval $[a, b]$ influence the solution in the area between the characteristic lines

$$x + ct = a \quad \text{and} \quad x - ct = b.$$

This area is called *the range of influence* of the interval $[a, b]$.

Example 1.4.1

Solve the IVP

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R},\ t \in \mathbb{R},$$

with IC,

$$\begin{aligned} u(x,0) &= f(x) = \begin{cases} h, & |x| \le a \\ 0, & |x| > a \end{cases} \\ u_t(x,0) &= 0. \end{aligned}$$

Solution: The d'Alembert solution is

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2},$$

since $u_t(x,0)=0$, and

$$f(x-ct) = \begin{cases} h, & |x-ct| \le a \\ 0, & |x-ct| > a \end{cases},$$
$$f(x+ct) = \begin{cases} h, & |x+ct| \le a \\ 0, & |x+ct| > a \end{cases}.$$

The simplest way of expressing the solution is by drawing it in the $x-t$ plane, see Figure 1.6. The initial condition is a square wave of

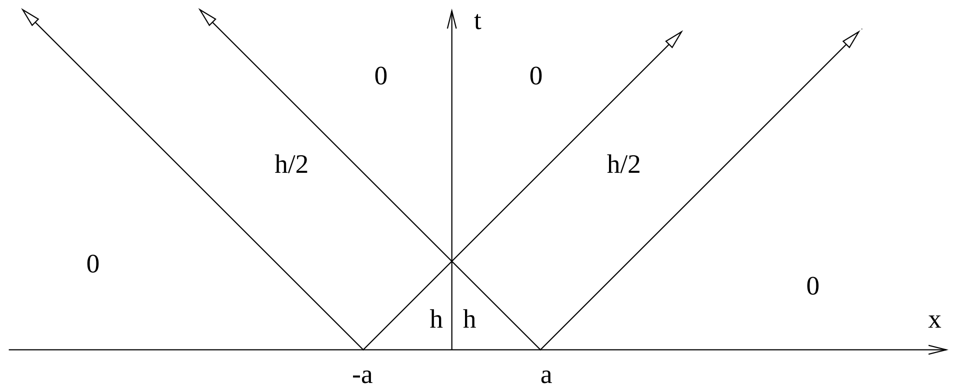

Figure 1.6: The d'Alembert solution with an initial box wave.

height h and base $2a$. This box splits into two boxes with base $2a$ but height $h/2$. One of the boxes propagates to the left, the other one to the right, see Figures 1.6 and 1.7. This means that the solution is zero except between the two characteristics on Figure 1.6. If we place an observer at a point $b > a$ on the positive x axis then initially he sees no wave. Then he sees a sharp wavefront and then a wave of height $h/2$, it last for a period of length $2a/c$ and then it ends sharply. Afterwards he sees no wave. This is a typical behavior of solutions to hyperbolic equations in odd dimensions. They do not smooth the initial data. (Parabolic and elliptic equation on the other hand make the initial data infinitely smooth.) The wave has a sharp arrival time and in odd dimension it also has a sharp departure time.

We now summarize the properties of the solutions of wave equations.

Property 1.4.2 *The waves propagate with finite speed.*

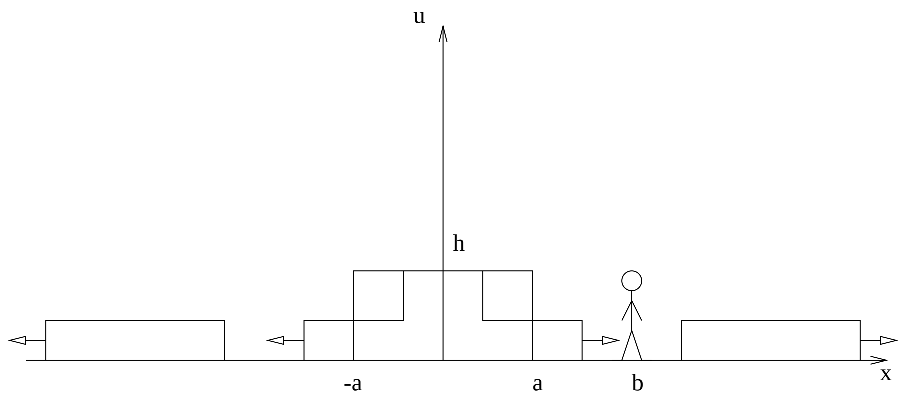

Figure 1.7: The splitting of the box wave.

Property 1.4.3 *Singularities and sharp corners of the initial data are not made smoother by the time-evolution.*

Property 1.4.4 *If damping terms are absent, such as in the wave equation, the solutions do not decay.*

Example 1.4.5

Solve the IVP

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R},\ t \in \mathbb{R},$$

with IC,

$$\begin{aligned} u(x,0) &= 0 \\ u_t(x,0) &= g(x) = \begin{cases} h, & |x| \leq a \\ 0, & |x| > a \end{cases} \end{aligned}$$

Solution: The d'Alembert solution is

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds,$$

since $u(x,0) = 0$, however this formula is misleading because g may vanish over a portion of the interval of integration. The best way to figure out what the limits of integration should be is to draw the domain of dependence for points in the six regions between and around the characteristics on Figure 1.8. We draw the triangle with vertex the point where the solution is to be determined and having the two characteristics through that point as sides. Then the intersections points with the initial line, marking the base of the triangle, see Figure 1.8, determine the limits of integration. The regions to the left of $-a$ and to the right of a just give 0 because $g = 0$ on the corresponding domain of dependence, the remaining regions give the limits of integration below, see Figure 1.8. The complete solution can now be labelled by the regions on Figure 1.8,

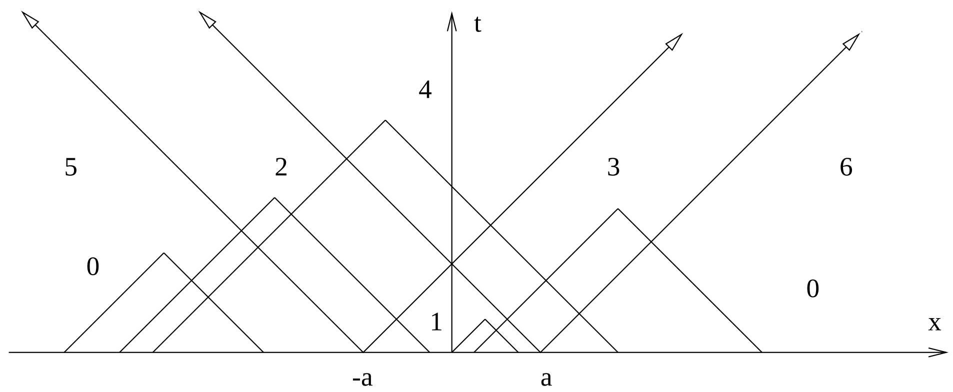

Figure 1.8: The domains of dependence.

1. $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} h\,ds = ht$
2. $u(x,t) = \frac{1}{2c}\int_{-a}^{x+ct} h\,ds = \frac{h}{2c}(x+ct+a)$
3. $u(x,t) = \frac{1}{2c}\int_{x-ct}^{a} h\,ds = \frac{h}{2c}(a-x+ct)$
4. $u(x,t) = \frac{1}{2c}\int_{-a}^{a} h\,ds = \frac{ha}{c}$
5. $u(x,t) = 0$
6. $u(x,t) = 0.$

Exercises 1.4.6

1.2 Find a solution of the IVP,

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+, \\ u(x,0) &= \begin{cases} 0 & |x| < a \\ h & |x| \geq a \end{cases} \\ u_t(x,0) &= 0. \end{aligned}$$

1.3 Find a solution of the IVP,

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+, \\ u(x,0) &= 0 \\ u_t(x,0) &= \begin{cases} 0 & |x| \leq a \\ h & |x| > a \end{cases} \end{aligned}$$

Draw the solutions in the $x-t$ plane.

1.5 The Semi-infinite String

We assume now that the string extends from its point of origin $x = 0$ to $x = \infty$. Then we have to specify how the string is behaving at the origin and this is called a boundary value (condition),

$$\text{BC:} \quad u(0,t) = h(t).$$

$h(t)$ describes how the end of the string moves and if we assume it is held tight, then $h = 0$ and we get an IVP with a boundary condition,

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}^+,\ t \in \mathbb{R}^+,$$

IC,

$$u(x,0) = f(x),\ u_t(x,0) = g(x), \quad x \in \mathbb{R}^+$$

BC,

$$u(0,t) = 0, \quad t \in \mathbb{R}^+.$$

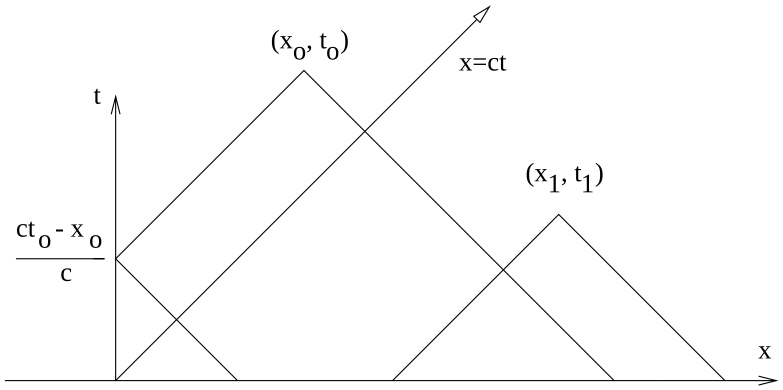

Figure 1.9: The reflection of the wave at the origin.

The simplest way of computing the solution is by considering the characteristic lines in the $x-t$ plane. A wave that starts at the origin propagates along the line $x = ct$, see Figure 1.9. If we select a point (x_1, t_1) below the line $x = ct$, then a wave starting at the origin has not had sufficient time to propagate to the point $x_1 > c\,t_1$. This is a consequence of Property 1.4.2, the propagation speed is $c < \infty$. The solution at the point (x_1, t_1) is completely determined by the d'Alembert solution, in Section 1.3, with initial data propagating along the characteristics $x + ct = x_1 + ct_1$ and $x - ct = x_1 - ct_1$, see Figure 1.9. The solution at (x_0, t_0) is on the other hand determined by data carried along the characteristics $x + ct = x_0 + ct_0$ and $x - ct = x_0 - ct_0$. The latter line never reaches the x axis but intersects the t axis at the point $(0, (ct_0 - x_0)/c)$. At this point the solution is determined by data that propagated along the line $x + ct = ct_0 - x_0$ and the boundary condition. Physically this is easy to understand, the initial data at every point on the x-axis splits into two waves one that propagates right and another one that propagates left, compare Figure 1.9. The left-going wave eventually reaches the end of the string and is reflected into a right-going wave. In the region $ct > x$, the solution is determined by a left-travelling wave and a right-travelling wave that is the reflection of a left-travelling wave.

We will now use the boundary conditions to determine how the wave is reflected,

$$u(0,t) = \phi(-ct) + \psi(ct) = 0$$

by the derivation in Section 1.3, or

$$\phi(-ct) = -\psi(ct).$$

We introduce a new coordinate $\xi = -ct$, then

$$\phi(\xi) = -\psi(-\xi).$$

This shows that for $-\xi > 0$, the value of ϕ on the negative x axis $\xi = x < 0$, is determined by the values of ψ on the positive $x = -\xi > 0$ axis. It is a little confusing that we do not let ξ be equal to x but we will see in a moment why. However, it is clear that the left-going wave ψ has determined the right-going wave ϕ. We now use the d'Alembert expression, from Section 1.3, in the last equation,

$$\phi(x-ct) = -\psi(ct-x) = -\frac{1}{2}f(ct-x) - \frac{1}{2c}\int_0^{ct-x} g(s)\,ds.$$

This makes it clear why we had to use $-\xi = ct - x > 0$, it is this expression that is positive in the region $ct > x$. Finally, by adding the left-going and right-going waves we obtain the complete solutions,

$$u(x,t) = \begin{cases} \frac{1}{2}[f(x+ct) + f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds, & x > ct \\ \frac{1}{2}[f(ct+x) - f(ct-x)] + \frac{1}{2c}\int_{ct-x}^{ct+x} g(s)ds, & x < ct. \end{cases} \tag{1.2}$$

If we shake the end of the string at $x = 0$, instead of keeping it fixed as above, then the boundary condition becomes

$$u(0,t) = h(t)$$

and

$$\phi(\xi) = -\psi(-\xi) + h(-\xi/c)$$

in the derivation above. This implies that we must add $h\left(t - \frac{x}{c}\right)$ to the latter part of the above formula 1.2.

Exercises 1.5.1

1.4 Find the solution of the IVP,

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \qquad x \in \mathbb{R}^+, t \in \mathbb{R}^+ \\ u(x,0) &= \begin{cases} h, & |x-2| \le 1 \\ 0, & |x-2| > 1 \end{cases} \\ u_t(x,0) &= 0. \end{aligned}$$

with the BC,

$$u_x(0,t) = 0.$$

Draw the solution in the $x - t$ plane.

1.5 Solve the IVP

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}^+, \; t \in \mathbb{R}^+,$$

IC,

$$u(x,0) = 0, \; u_t(x,0) = \begin{cases} 0, & x \le a \\ 1, & x > a \end{cases}$$

BC,

$$u_x(0,t) - bu(0,t) = 0, \; t \in \mathbb{R}^+.$$

Draw the solution in the $x - t$ plane.

1.6 The Inhomogenous Wave Equation

We now solve the inhomogenous wave equation

$$u_{tt} - c^2 u_{xx} = F(x,t), \quad x \in \mathbb{R}, \; t \in \mathbb{R},$$

IC,

$$u(x,0) = 0, \; u_t(x,0) = 0.$$

We use characteristics and obtain, just as when we solved the wave equation in Section 1.3,

$$-4cu_{\xi\eta} = F(\xi, \eta),$$

where

$$\xi = x - ct, \quad \eta = x + ct,$$

are the characteristic coordinates. This equation is then integrated twice with respect to ξ and η, but first we have to figure out what the initial data should be as functions of ξ and η. The initial line $t = 0$ is the line $\xi = \eta$, see Figure 1.10.

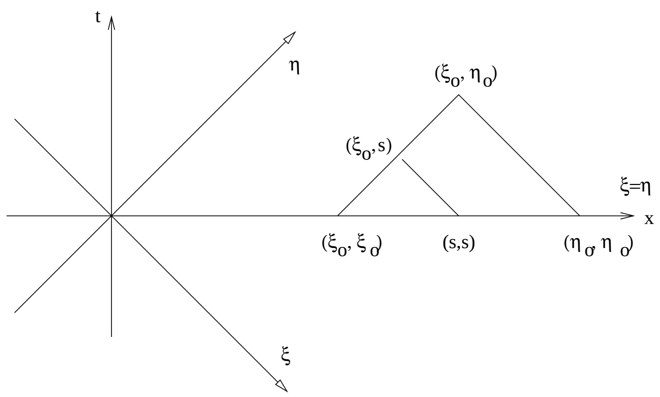

Figure 1.10: The characteristic cone.

The first initial condition is unchanged

$$u(x, 0) = u(\xi, \eta) = 0, \quad \xi = \eta,$$

since the equation holds for all x, we differentiate with respect to x and conclude that

$$u_x(x, 0) = 0.$$

We recall the expressions for the u_t and u_x derivatives in Section 1.3, and solve these for the u_ξ and u_η derivatives,

$$u_\xi = u_x - cu_t \quad \text{and} \quad u_\eta = u_x + cu_t.$$

Therefore, if

$$u_x(x, 0) = 0 \text{ and } u_t(x, 0) = 0, \quad \text{on } t = 0,$$

then

$$u_\xi(\xi, \eta) = 0 \text{ and } u_\eta(\xi, \eta) = 0, \quad \text{on } \xi = \eta.$$

Consequently, we can either take the initial conditions

$$u(\xi,\xi)=0,\ \ u_\xi(\xi,\xi)=0,$$

or

$$u(\eta,\eta)=0,\ \ u_\eta(\eta,\eta)=0.$$

They are equivalent.

Now we integrate the equation

$$u_{\xi\eta}=-\frac{1}{4c^2}F(\xi,\eta)$$

with respect to ξ from a point (ξ,s) to the point (s,s), on the initial line $\xi=\eta$, see Figure 1.10. We obtain

$$u_\eta(s,s)-u_\eta(\xi_0,s)=-u_\eta(\xi_0,s)$$

since by the IC, $u_\eta(s,s)=0$,

$$=-\frac{1}{4c^2}\int_s^{\xi_0}F(r,s)\,ds.$$

Next we integrate with respect to η from (ξ_0,ξ_0) to the point (ξ_0,η_0), which also lies on the initial line, see Figure 1.10. Then we obtain

$$\begin{aligned}-u(\xi_0,\eta_0)+u(\xi_0,\xi_0) &= -u(\xi_0,\eta_0)\\ &= -\frac{1}{4c^2}\int_{\xi_0}^{\eta_0}\int_s^{\xi_0}F(r,s)dsds,\end{aligned}$$

by use of the other IC, $u(\xi,\xi)=0$. This gives the formula

$$u(\xi,\eta)=\frac{1}{4c^2}\int_\xi^\eta\int_\xi^s F(r,s)drds,$$

where the double integral is over the triangle in Figure 1.10, with corners (ξ,ξ), (ξ,η) and (η,η). We can also integrate over this triangle by integrating over the line

$$x-c(t-\tau)\leq y\leq x+c(t-\tau),$$

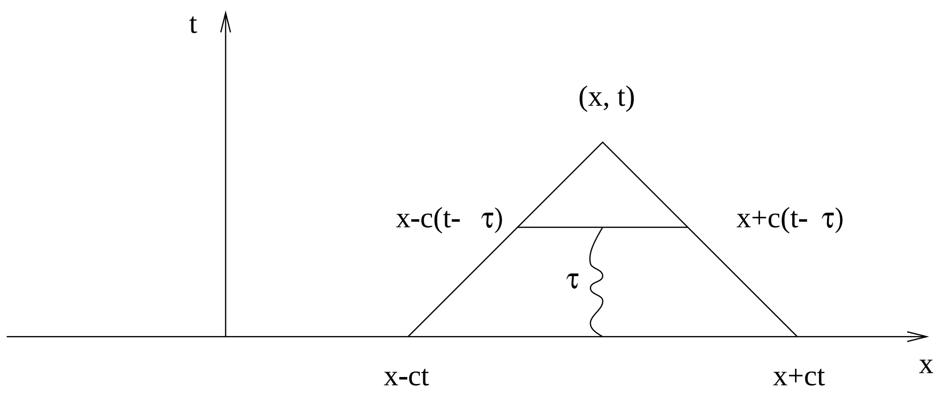

Figure 1.11: The characteristic cone in rectangular coordinates.

see Figure 1.11, and then integrating τ from 0 to t. The Jacobian of the change of variables from (ξ, η) to (x, t) is

$$\frac{\partial(\xi,\eta)}{\partial(x,t)} = \det\begin{pmatrix} 1 & -c \\ 1 & c \end{pmatrix} = 2c,$$

so

$$d\xi\, d\eta = 2c\, dx\, dt.$$

This gives the solution

$$u(x,t) = \frac{1}{2c}\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)} F(y,\tau)\, dy\, d\tau. \qquad (1.3)$$

Exercises 1.6.1

1.6 Solve the IVP,

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= e^x, \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+ \\ u(x,0) &= 0 \\ u_t(x,0) &= 0. \end{aligned}$$

1.7 Solve the IVP,

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= \cos x, \qquad x \in \mathbb{R}^+,\ t \in \mathbb{R}^+ \\ u(x,0) &= 0 \\ u_t(x,0) &= 0. \end{aligned}$$

with the BC,

$$u(0,t) = \sin(t).$$

Chapter 2

Superposition and Weak Solutions

2.1 The Superposition Principle

A linear partial differential equation (PDE) can be thought of as a linear partial differential operator $P(D)$ acting on the solution u, namely

$$P(D)u = 0.$$

For example,

$$P(D) = \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2},$$

is the operator for the wave equation. The superposition principle says that $P(D)$ acts linearly, or if $u_1, u_2, \ldots, u_n$ are all solutions, corresponding to initial data, $f_1, f_2, \ldots, f_n$, then $u_1 + \ldots + u_n$ is also a solution, with initial data $f_1 + \ldots + f_n$. Namely, for two solutions

$$P(D)(u_1 + u_2) = P(D)u_1 + P(D)u_2 = 0,$$

by the linearity of P, since u_1 and u_2 are solutions. Thus $u_1 + u_2$ is also a solution. For three solutions, we can add u_3 to $u_1 + u_2$ etc. Suppose the PDE is an nth order equation for which an initial value problem is appropriate, (the wave equation is a second order equation) then

$$(u_1 + u_2)^{(j)}(0) = u_1^{(j)}(0) + u_2^{(j)}(0) = f_j(1) + f_j(2),\ 1 \leq j \leq n,$$

where $u^{(j)}$ denotes the jth derivative in the direction of the normal to the initial surface. (This is the t-direction for the wave equation.) Boundary conditions are handled in a similar manner and this give us the following principle. We state it for just two solutions but it holds for n solutions by the above argument (or induction).

The Superposition Principle 2.1.1 *If u_1 is a solution of a linear nth order PDE with initial data $(f_1(1), f_2(1), \cdots, f_n(1))$ and u_2 is a solution of the same PDE with initial data $(f_1(2), \cdots, f_n(2))$, then $u_1 + u_2$ is also a solution with initial data $(f_1(1) + f_1(2), \cdots, f_n(1) + f_n(2))$.*

The superposition principle can also be used to decompose an inhomogeneous PDE into a homogeneous PDE with all the initial data, and an inhomogeneous PDE with initial data zero. If $w = u + v$, then

$$P(D)w = P(D)u + P(D)v = F$$

and this give the principle.

The Superposition Principle 2.1.2 *If u is a solution of the homogeneous IVP, $P(D)u = 0,\ u(0) = f,\ u_t(0) = g$, and v is a solution of the inhomogeneous IVP, $P(D)v = F,\ v(0) = 0,\ v_t(0) = 0$, then $w = u+v$, is a solution of the IVP, $P(D)w = F,\ w(0) = f,\ w_t(0) = g$.*

The function w is a solution of the PDE by the above and we just have to check that it satisfies the initial conditions,

$$w(0) = u(0) + v(0) = f, \quad w_t(0) = u_t(0) + v_t(0) = g.$$

In particular the superposition principle 2.1.2 implies for the wave equation, that the solution of the imhomogeneous IVP,

$$u_{tt} - c^2 u_{xx} = F(x,t), \quad x \in \mathbb{R},\ t \in \mathbb{R},$$

IC,

$$u(x,0) = f(x),\ u_t(x,0) = g(x),$$

is

$$\begin{aligned} u(x,t) &= \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds \\ &+ \frac{1}{2c}\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)} F(y,\tau)\,dy\,d\tau. \end{aligned} \tag{2.1}$$

Example 2.1.3

Solve the IVP,

$$u_{tt} - c^2 u_{xx} = \sin(x+t), \quad x \in \mathbb{R},\ t \in \mathbb{R},$$

IC,

$$u(x,0) = 0, \quad u_t(x,0) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1. \end{cases}$$

Solution: We use the superposition principle and solve first the homogeneous IVP,

$$\begin{gathered} u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \\ u(x,0) = 0, \quad u_t(x,0) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1. \end{cases} \end{gathered}$$

It has the D'Alembert solution 1.1, from Section 1.3,

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds = \frac{h(x+ct)-h(x-ct)}{2c},$$

where

$$\begin{aligned} h(x+ct) &= \begin{cases} x+ct, & |x+ct| \le 1 \\ 0, & |x-ct| > 1 \end{cases} \\ h(x-ct) &= \begin{cases} x-ct, & |x-ct| \le 1 \\ 0, & |x-ct| > 1, \end{cases} \end{aligned}$$

see Figure 2.1. The limits of integration in the d'Alembert formula are found by drawing the two characteristics from each point in the $x-t$

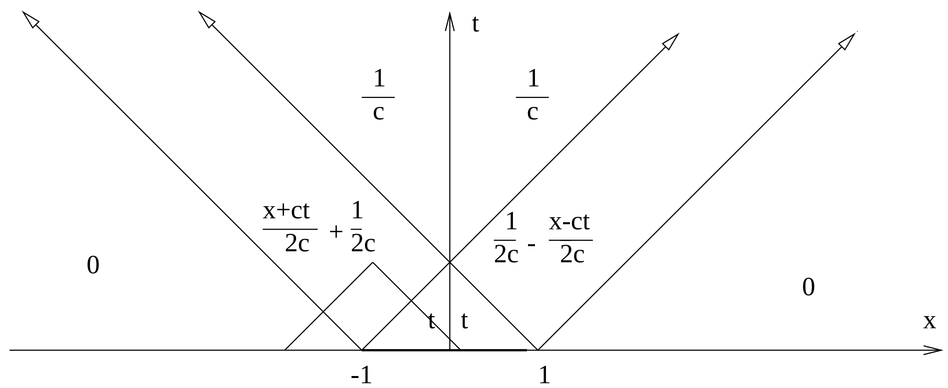

Figure 2.1: The homogeneous part of the solution.

plane down to the x axis and integrating over the part of the domain of dependence, see Example 1.4.5, that overlaps the interval $[-1,1]$. Then we solve the homogeneous IVP with zero initial conditions,

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= \sin(x+t), \quad x \in \mathbb{R},\ t \in \mathbb{R}, \\ u(x,0) &= 0 = u_t(x,0). \end{aligned}$$

The solution of inhomogeneous wave equation in Section 1.6 gives us the solution,

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \sin(r+\tau)\,dr\,d\tau \\ &= -\frac{1}{2c}\int_0^t [\cos(x+ct+(1-c)\tau) - \cos(x-ct+(1+c)\tau)]\,dt \\ &= -\frac{1}{2c}\left[\sin(x+ct+(1-c)\tau)\Big|_{\tau=0}^{\tau=t} - \sin(x-ct+(1+c)\tau\Big|_{\tau=0}^{\tau=t}\right] \\ &= -\frac{1}{2c}\left[\frac{\sin(x+t)-\sin(x+ct)}{1-c} - \frac{\sin(x+t)-\sin(x-ct)}{1+c}\right] \\ &= \frac{\sin(x+ct)}{c(1-c)} - \frac{\sin(x-ct)}{c(1+c)} - \frac{\sin(x+t)}{(1-c^2)}. \end{aligned}$$

Then finally by the superposition principle the solution to the original inhomogeneous IVP, with non-vanishing initial data is:

$$u(x,t) = \frac{h(x+ct) - h(x-ct)}{2c} + \frac{\sin(x+ct)}{c(1-c)} - \frac{\sin(x-ct)}{c(1+c)} - \frac{\sin(x+t)}{1-c^2},$$

where the values of h are given by the above formula and are shown in Figure 2.1.

Exercises 2.1.4

2.1 Find the solution of the IVP,

$$u_{tt} = u_{xx}, \qquad x \in \mathbb{R}^+,\ t \in \mathbb{R}^+$$

IC,

$$u(x,0) = x, \quad u_t(x,0) = 3x^2, \quad x \in \mathbb{R}^+,$$

BC,

$$u_x(0,t) = 0, \quad t \in \mathbb{R}^+.$$

Draw the solution in the $x - t$ plane.

2.2 Solve the IVP,

$$u_{tt} = u_{xx} + x, \quad x \in \mathbb{R},\ t \in \mathbb{R}^+,$$

IC,

$$u(x,0) = \begin{cases} 1, & 0 \le 0 \le 1, \\ 0, & \text{otherwise} \end{cases}, \quad u_t(x,0) = 0,\ x \in \mathbb{R}.$$

Draw the solution in the $x - t$ plane.

2.2 The Wave Equation on a Finite Interval

A violin string, discussed in Section 1.2, has finite length ℓ and its ends are tied to the violin. This means that it satisfies an initial value problem with boundary conditions (BC),

$$u_{tt} = c^2 u_{xx}, \quad 0 \leq x \leq \ell, \ \ t \in \mathbb{R}^+,$$

IC,

$$u(x,0) = f(x), \ u_t(x,0) = g(x),$$

BC,

$$u(0,t) = 0 = u(\ell,t).$$

The boundary conditions say that the ends of the string at $x = 0$ and

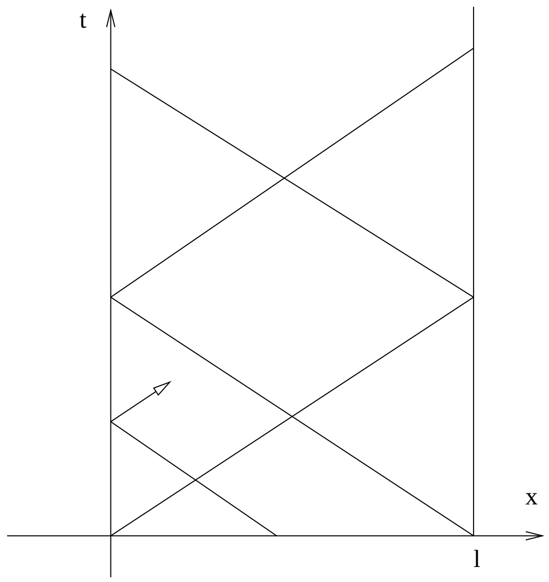

Figure 2.2: The characteristics on a finite interval.

$x = \ell$ are tied to the ends of the violin at all times.

Now consider Figure 2.2. If we are at a point (x,t) whose domain of dependence lies within the interval $[0,\ell]$, these points form the bottom triangle on Figure 2.2, then the solution is the usual d'Alembert solution from Section 1.3,

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

Outside the bottom triangle on Figure 2.2 we have to extend the initial data to the negative x axis, as we did for the semi-infinite string in Section 1.5. This means that we have to substitute the general solution

$$u(x,t) = \phi(x-ct) + t(x+ct)$$

into the BC, at $x = 0$, to conclude that

$$\phi(-x) = -\psi(x),$$

see Figure 2.2. Thus

$$\frac{f(-x)}{2} + \frac{1}{2c}\int_0^{-x} g(s)\,ds = -\frac{f(x)}{2} - \frac{1}{2c}\int_0^{x} g(s)\,ds$$

as we must extend f and g as odd functions to the interval $[-\ell, 0]$,

$$f(-x) = -f(x), \quad g(-x) = -g(x), \quad x \in [0,\ell],$$

see Figure 2.3. By the same argument

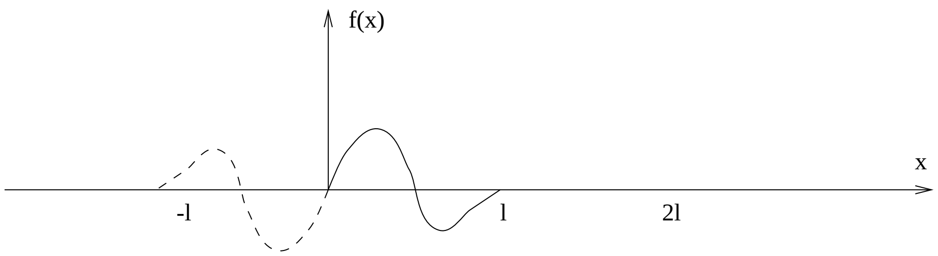

Figure 2.3: The odd extension of f.

$$\psi(x+\ell) = -\phi(\ell - x)$$

using the BC, at $x = \ell$, and

$$\frac{f(\ell+x)}{2} + \frac{1}{2c}\int_{\ell}^{\ell+x} g(s)\,ds = -\frac{f(\ell-x)}{2} - \frac{1}{2c}\int_{\ell-x}^{\ell} g(s)\,ds,$$

if

$$f(\ell+x) = -f(\ell-x), \quad g(\ell+x) = -g(\ell-x).$$

This shows that we must also extend f and g as odd functions, about $x = \ell$, to the interval $[\ell, 2\ell]$.

We continue this argument to extend f and g as odd periodic function with period 2ℓ to the whole x axis, then the d'Alembert formula determines the solution

$$u(x,t) = \frac{f^*(x+ct) + f^*(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\,ds, \tag{2.2}$$

where * denotes the odd extensions of f and g to $\mathbb{R}$.

The tied boundary conditions made it necessary to extend the initial data f and g as odd periodic functions with period $2l$. Different boundary conditions will require different extensions. If the ends of the violin string move freely on a rod, these are called free boundary conditions,

$$u_x(0,t) = 0 = u_x(l,t) \quad t \in \mathbb{R},$$

then we must extend f and g, about 0 and l, as even periodic functions with period $2l$, see Figure 2.4.

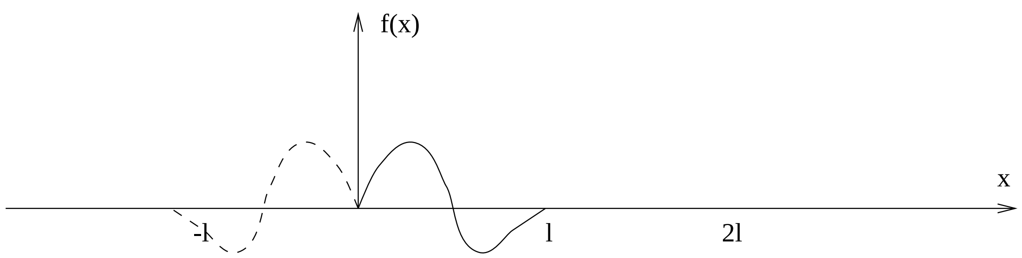

Figure 2.4: The even extension of f.

Example 2.2.1

Solve the IVP,

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 \leq x \leq \ell,\ t \in \mathbb{R}^+, \\ u(x,0) &= 1, \quad 0 \leq x < \ell, \\ u_t(x,0) &= 0, \end{aligned}$$

with tied (Dirichlet) BC,

$$u(0,t) = 0 = u(\ell,t).$$

Solution: We extend $u(x,0) = f(x)$ as an odd periodic function with period 2ℓ to the whole x axis,

$$f^* = \begin{cases} 1, & 2n\ell \leq x \leq (2n+1)\ell \\ -1, & (2n+1)\ell \leq x \leq (2n+2)\ell, \end{cases} \qquad n \in \mathbb{Z},$$

then u is given by the d'Alembert solution, see Figure 2.5. These boundary conditions are the same as if waves were impinging on and being reflected by a hard wall.

Exercises 2.2.2

2.3 A violin string has length ℓ and the initial shape $u = \sin(\pi x/\ell)$ when it is let go from rest. Determine the oscillations of the string thereafter.

2.4 Determine the oscillations of the string if it is initially flat but is pushed with the initial velocity $u_t = \cos(\pi x/\ell)$.

2.5 Solve the IVP and draw the solution in the $x - t$ plane,

$$u_{tt} = c^2 u_{xx}, \quad 0 \leq x \leq 3,\ t \in \mathbb{R}^+,$$

IC,

$$u(x,0) = 0,\ u_t(x,0) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \\ 0, & 2 \leq x \leq 3 \end{cases}.$$

BC,

$$u(0,t) = 0 = u(3,t),\ t \in \mathbb{R}^+.$$

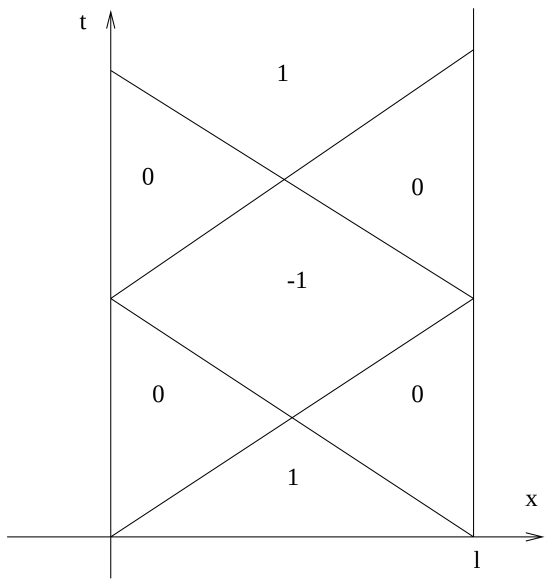

Figure 2.5: The waves reflected by a hard wall.

2.6 Solve the IVP,

$$u_{tt} = c^2 u_{xx} + 16x^2, \quad 0 \le x \le L,\ t \in \mathbb{R}^+,$$

IC,

$$u(x,0) = x(L-x),\ u_t(x,0) = 0,$$

BC,

$$u(0,t) = 0 = u(L,t),\ t \in \mathbb{R}^+.$$

Hint: Express the inhomogeneity in terms of characteristic coordinates,

$$x^2 = [(x+ct)^2 + (x-ct)^2]/4.$$

2.3 Well-posedness, Weak Solutions and Distributions

We now ask what conditions the initial data f and g, from Section 1.3, must satisfy so that the d'Alembert formula gives us a solution of the wave equation. It is natural to assume that f and g can be differentiated because we must differentiate them in order to show that the d'Alembert formula satisfies the wave equation.

Theorem 2.3.1 *If $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$, i.e. f has two continous derivatives and g one and $F(x,t) \in C^1(\mathbb{R} \times \mathbb{R})$, so that F is continously differentiable both in x and t, then*

$$\begin{aligned} u(x,t) &= \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds \\ &\quad + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(s,\tau)\,ds\,d\tau, \end{aligned}$$

is the unique solution of the IVP,

$$u_{tt} - c^2 u_{xx} = F(x,t), \quad x \in \mathbb{R},\ t \in \mathbb{R},$$

IC,

$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

Proof: It is clear from the formula for u and the conditions on f and g that u is twice differentiable in x and t. We differentiate $u(x,t)$ with respect to t and x,

$$\begin{aligned} u_t &= c\left[\frac{f'(x+ct)-f'(x-ct)}{2}\right] + \frac{g(x+ct)-g(x-ct)}{2} \\ &\quad + \frac{1}{2}\int_0^t [F(x+c(t-\tau),\tau) + F(x-c(t-\tau),\tau)]\,d\tau, \\ u_{tt} &= c^2\left[\frac{f''(x+ct)+f''(x-ct)}{2}\right] + c\left[\frac{g'(x+ct)-g'(x-ct)}{2}\right] \\ &\quad + \frac{F(x,t)+F(x,t)}{2} + \frac{c}{2}\int_0^t [F'(x+c(t-\tau),\tau) \\ &\quad - F'(x-c(t-\tau),\tau)]\,d\tau, \\ u_x &= \frac{f'(x+ct)+f'(x-ct)}{2} + \frac{g(x+ct)-g(x-ct)}{2c} \end{aligned}$$

$$+\frac{1}{2c}\int_0^t [F(x+c(t-\tau),\tau)-F(x-c(t-\tau,\tau)]\,d\tau,$$

$$u_{xx} = \frac{f''(x+ct)+f''(x-ct)}{2}+\frac{g'(x+ct)-g'(x-ct)}{2c}$$
$$+\frac{1}{2c}\int_0^t [F_x(x+c(t-\tau),\tau)-F_x(x-c(t-\tau),\tau)]\,d\tau.$$

Thus

$$u_{tt}-c^2u_{xx}=F(x,t).$$

The uniqueness follows from the fact that if $f \in C^2$, $g \in C^1$ and $F \in C^1(\mathbb{R}\times\mathbb{R})$ then all the operations we used in deriving the solution of the inhomogeneous wave equation, in Sections 1.3 and 1.6, are mathematically justified and they give a unique formula for the solution
$\square$

Remark 2.3.2 *Theorem 2.3.1 gives us existence and uniquness of the solutions. If they are also stable, or small perturbations of the initial data result in solutions that remain close to the unpertubed solutions, then the IVP is said to be well-posed. If the IVP is well-posed, one becomes more confident that it gives a correct description of reality. If it is ill-posed, or any one of the three conditions above does not hold, then the IVP may not be a good description of physical reality.*

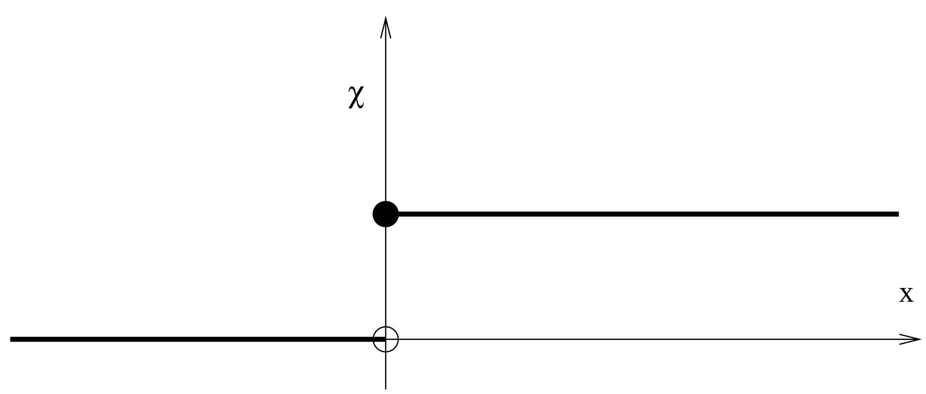

Figure 2.6: The Heavyside function.

Theorem 2.3.1 does not apply to the IVP's in Examples 1.1, 1.2 and some of the exercises. The reason is that the initial data, f and g, in these problems does not have enough derivatives. In fact, these data are simple examples of generalized functions called distributions. The simplest one of these is called the Heavyside function, see Figure 2.6.

$$\chi(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

The Dirac delta function, see Figure 2.7, is defined to be the derivative

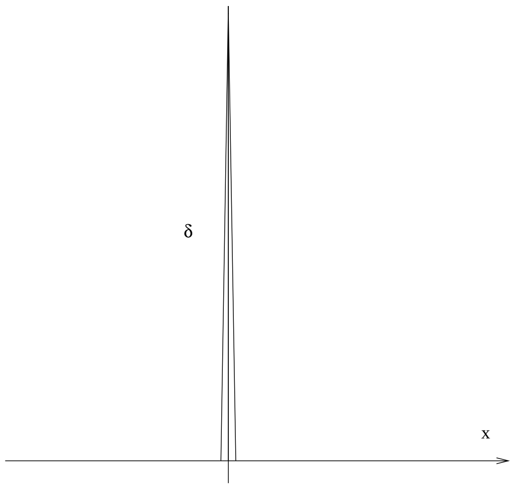

Figure 2.7: The Dirac delta function.

of the Heavyside function,

$$\delta(x) = \frac{d}{dx}\chi(x),$$

and then we can define the derivative of the delta function

$$\delta'(x) = \frac{d}{dx}\delta(x),$$

etc. Of course none of these derivatives exist as functions but they are distributions and in particular the delta function is zero except at the origin, where it has an infinitely high and narrow peak, see Figure 2.7.

δ' on the other hand has two peaks one positive at 0^-, just below 0, and one negative at 0^+, just above 0, see Figure 2.8.

The δ function has the property that it picks out the value of a continuous function at zero, when integrated against it,

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0).$$

The derivatives of δ pick out the derivatives of f,

$$\int_{-\infty}^{\infty} f(x)\delta^{(k)}(x)\,dx = (-1)^k f^{(k)}(0),$$

where $f^{(k)}(x) = \frac{d^k}{dx^k} f(x)$. Now let us deduce the above identity for the δ function; since $\delta = 0$, for $x < 0$ and $x > 0$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\delta(x)\,dx &= \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} f(x)\delta(x)\,dx \\
&= \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} f(x)\chi'(x)\,dx \\
&= \lim_{\epsilon\to 0} f(x)\chi(x)\Big|_{-\epsilon}^{\epsilon} - \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} f'(x)\chi(x)\,dx,
\end{aligned}$$

by integration by parts. Thus

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = \lim_{\epsilon\to 0} f(\epsilon)\chi(\epsilon) = f(0),$$

because $\chi = 0$, for $x < 0$, so $\chi(-\epsilon) = 0$, and

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} f'(x)\chi(x)\,dx = 0,$$

assuming f' is continuous, since χ is continuous except at $x = 0$.

If we translate the delta function $\delta(x + x_0)$, then the singularity simply moves to $x = -x_0$. The initial data in Example 1.1 can be expressed in terms of a translated Heavyside functions. Namely,

$$f(x) = \begin{cases} h, & -a \le x < a \\ 0, & x < -a,\ x \ge a, \end{cases}$$

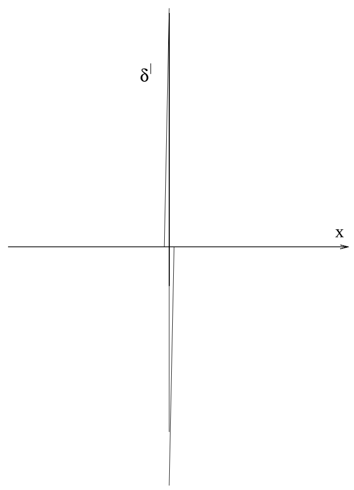

Figure 2.8: The derivative of the delta function.

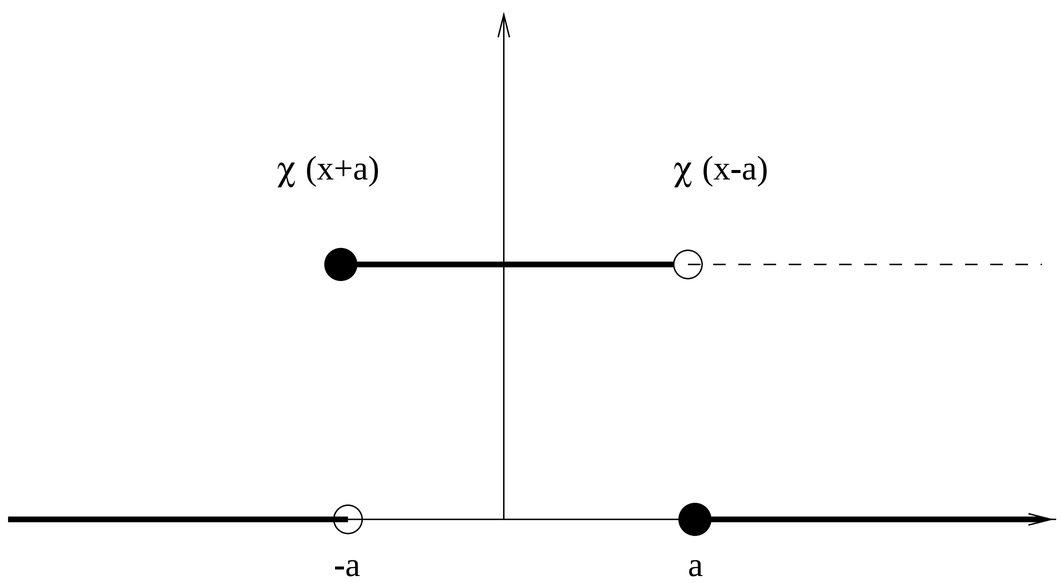

Figure 2.9: The Heavyside box function.

can be expressed as

$$f(x) = h[(\chi(x+a) - \chi(x-a)],$$

see Figure 2.9. Notice, however, that the limit from the left of the function, at $x = a$, is missing in this formulation. This makes it explicit what the properties of d'Alembert solution are

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2},$$

assuming $g = u_t(x,0) = 0$. *This is not even a continuous function* and does certainly not have the two derivatives required for the application of Theorem 2.3.1.

We are now faced with the question: What do we mean when we say that such a discontinuous u, as above, is a solution of the wave equation? The correct interpretation is that u is a *weak* solution of the PDE.

Definition 2.3.3 A function $u(x,t)$ is a weak solution of a PDE: $P(\partial_t, \partial_x) = 0$, if for every test function $\varphi(x,t) \in C_0^\infty(\mathbb{R} \times \mathbb{R})$,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u P(\partial_t, \partial_x) \varphi \, dx \, dt = 0.$$

The notation C^∞ indicates that φ, is a *smooth* function of its variables, or has infinitely many derivatives. The subscript 0 in C_0 indicates that φ vanishes outside some finite region in the $x - t$ plane.

The important point to notice here is that the PDE acts on the test function φ which has infinitely many derivatives both in x and t, but not on the weak solution u which may not have any derivatives. Moreover, the integral has to vanish for all such smooth φ, for u to be a weak solution. For the wave equation, we get that u is a weak solution if

$$\int_{-\infty}^{\infty} \int_{\infty}^{\infty} u(x,t)(\varphi_{tt} - c^2 \varphi_{xx}) \, dx \, dt = 0,$$

for all $\varphi(x,t) \in C_0^\infty(\mathbb{R} \times \mathbb{R})$.

Example 2.3.4

Solve the IVP,

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+ \\ u(x,0) &= \delta(x-1) \\ u_t(x,0) &= \delta(x+1), \qquad x \in \mathbb{R}. \end{aligned}$$

Solution: By the d'Alembert formula

$$\begin{aligned} u(x,t) &= \frac{\delta(x-ct-1)+\delta(x+ct-1)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \delta(s+1)\, ds \\ &= \frac{\delta(x-ct-1)+\delta(x+ct-1)}{2} + \frac{1}{2c}, \end{aligned}$$

if $x-ct < -1 < x+ct$, and

$$u(x,t) = \frac{\delta(x-ct-1)+\delta(x+ct-1)}{2},$$

if $-1 < x-ct < 1$, see Figure 2.10, if $x+ct < -1$ or $x-ct > 1$, then $u(x,t) = 0$. We now show that this is a weak solution,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{\delta(x-ct-1)+\delta(x+ct-1)}{2} + \frac{1}{2c}\right)(\varphi_{tt} - c^2\varphi_{xx})dxdt$$

$$= -4c^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\delta(\xi-1)+\delta(\eta-1)}{2}\varphi_{\xi\eta} d\xi d\eta/2c,$$

since the integral of $\frac{1}{2c}$ vanishes, by the vanishing of φ at $\pm\infty$, and we have written the wave equation in characteristic coordinates, see Section 1.6. Recall from Section 1.6 that $2cdxdt = d\xi d\eta$. Now $\varphi_{\xi\eta}$ is a continous function at $x = 1$ so by the formula above for the action of the delta function on a continous function, we get,

$$= -4c\int_{-\infty}^{\infty} \varphi_{\xi\eta}(1,\eta)d\eta - 4c\int_{-\infty}^{\infty} \varphi_{\xi\eta}(\xi,1)d\xi$$

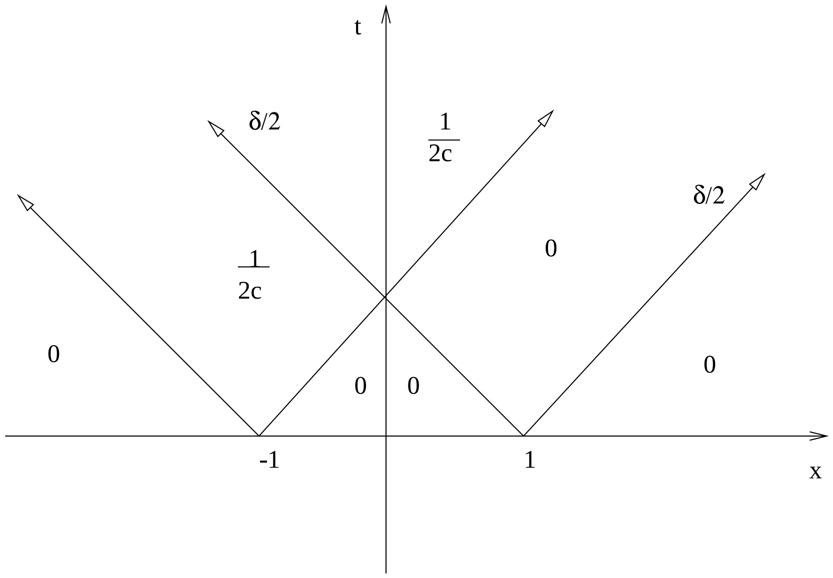

Figure 2.10: The weak solution.

$$= -4c\varphi_\xi(1,\eta)|_{-\infty}^{\infty} - 4c\varphi_\eta(\xi,1)|_{-\infty}^{\infty}$$
$$= 0,$$

since φ vanishes outside a finite region in the $\eta - \xi$ plane. This shows that u is a weak solution.

Exercises 2.3.5

2.7 Derive the identity

$$\int_{-\infty}^{\infty} f(x)\delta'(x)\,dx = -f'(x).$$

2.8 Find the weak solution of the IVP,

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R},\ t \in \mathbb{R}^+,$$

IC,

$$u(x,0) = 3\delta'(x-2), \quad u_t(x,0) = 4\delta(x+2), \quad x \in \mathbb{R},$$

where $\prime = d/dx$. Find and draw the path of the singularities in the $x - t$ plane.

2.9 Show that if the partial derivative $\frac{\partial u}{\partial x}$ of the solution u of the wave equation is discontinuous in some point (x_0, t_0), then the discontinuity propagates along the characteristic lines through (x_0, t_0). Find the jump in the derivatives

$$(\partial_t + c\partial_x)u \quad \text{and} \quad (\partial_t - c\partial_x)u,$$

if

$$\lim_{x \to x_0^+} \frac{\partial u}{\partial x}(x, t_0) - \lim_{x \to x_0^-} \frac{\partial u}{\partial x}(x, t_0) = 1,$$

but $\frac{\partial u}{\partial t}(x_0, t_0)$ is continuous.

Exercises for Chapters 1 and 2.

2.10 Vanishing (Dirichlet) BC require that the initial data f and $\int_0^x g(s)ds$ and the forcing F are extended from an interval $0 \leq x \leq \ell$, to the whole x-axis, as *odd* periodic functions with period 2ℓ. Free (Neumann) BC require an extension as even periodic functions with period 2ℓ. Solve the IVP,

$$u_{tt} - c^2 u_{xx} = 1 - x, \quad 0 \leq x \leq 1,\ t \in \mathbb{R}^+,$$

IC,

$$u(x, 0) = x^2(1 - x), \quad u_t(x, 0) = 0,\ 1 \leq x \leq 1,$$

BC,

$$u(0, t) = 0 = u_x(1, t), \quad t \in \mathbb{R}^+.$$

Is the solution weak or strong?

2.11 An infinite string has the initial shape e^{-x^2} and velocity xe^{-x^2} when it is let go. Pose the IVP for the oscillations of the string and show that the solution is *stable*, i.e. if we make a small errors ϵe^{-x^2} and $\epsilon x e^{-x^2}$, ϵ small, when measuring the initial shape and velocity, we still get a solution that is *close* to the correct solution.

Chapter 3

The Heat Equation

3.1 Derivation of the Heat Equation

Newton's law of cooling says that the heat that flows from a warm place to a colder one, during the time interval Δt, in materials, is

$$-K\frac{\partial u}{\partial x}\Delta t.$$

Here $u(x,t)$ is the temperature and K is the heat conductivity of the material. Newton's law says that the heat-flow is proportional to the gradient $\frac{\partial u}{\partial x}$ and the negative signature means that the flow is from the hotter place to the colder one. But the heat-flow out of a thin rod of material of length $b-a$ is also

$$S\frac{\partial u}{\partial t}(b-a)\Delta t,$$

where S is the specific heat of the material. This implies that

$$-S\frac{\partial u}{\partial t}(b-a) = -K\frac{\partial u}{\partial x}\bigg|_a^b.$$

This equation says that the decrease in heat, of the short rod of material, is proportional to the flow of heat out of the ends $x=a$ and $x=b$, of this one-dimensional rod. We divide by $S(b-a)$,

$$\frac{\partial u}{\partial t} = \frac{K}{S}\frac{\frac{\partial u}{\partial x}(b) - \frac{\partial u}{\partial x}(a)}{b-a},$$

and take the limit as $b \to a$. The resulting equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad \alpha = \frac{K}{S},$$

is called the heat equation, α is the heat diffusivity. The solution of the equation describes the heat distribution in the material as a function of time and the distance from the left end of the rod. The evolution of the temperature distribution is an example of diffusion, the random motion of molecules that distributes the heat. In the derivation of the equation we assumed that K and S are constants, this is valid for temperatures lying in some interval, and that the evolution towards a temperature equilibrium is slow compared to the molecular motion.

Heat is a concept that we have an intuition about from our daily lives. This intuition tells us that heat decreases as a function of time and if a body of some material is at a temperature equilibrium with its surroundings, then it will remain so, unless it is either heated or cooled. It is clear that it is important to give the initial temperature distribution, so it is not surprising that the appropriate problem for the heat equation is an initial value problem.

3.2 Solution of the Heat Equation, on the Line

We pose the IVP for the heat equation,

$$u_t = \frac{1}{2} u_{xx}, \quad x \in \mathbb{R},\ t \in \mathbb{R}^+,$$

IC,

$$u(x,0) = f(x), \quad x \in \mathbb{R},$$

where we have chosen the conductivity $\alpha = 1/2$ to simplify the computations.

Remark 3.2.1 *If $u(x,t)$ is a solution of*

$$u_t = \frac{1}{2} u_{xx},$$

then $u(x, 2\alpha t)$ is a solution of

$$u_t = \alpha u_{xx}.$$

This initial value problem describes the temperature $u(x,t)$ in an infinite rod as a function of position and time, $f(x)$ is the initial temperature. We will use the Fourier transform to solve the equation. The Fourier transform of a function $f(x)$ is

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}\, dx, \quad k \in \mathbb{R},$$

and the inverse Fourier transform of a function $g(k)$ is

$$\check{g}(x) = \int_{-\infty}^{\infty} g(k)e^{2\pi ikx}\, dk, \quad x \in \mathbb{R}.$$

We have assumed that both f and g are integrable functions. In fact if we can make sense of both $\hat{f}$ and $\check{\hat{f}}$ then

$$f(x) = (\check{\hat{f}}(k))(x).$$

Now we Fourier transform both sides of the heat equation,

$$\hat{u}_t = \frac{1}{2}\hat{u}_{xx} = \frac{1}{2}\int_{-\infty}^{\infty} u_{xx}(x)e^{-2\pi ikx}\, dx = -2\pi^2k^2\hat{u},$$

by integration by parts, where we have (formally) set the boundary terms equal to zero. This changes the PDE to an ODE

$$\hat{u}_t = -2\pi^2k^2\hat{u},$$

with the solution

$$\hat{u}(k,t) = \hat{u}(k,0)e^{-2\pi^2k^2t},$$

where

$$\hat{u}(k,0) = \int_{-\infty}^{\infty} u(x,0)e^{-2\pi ikx}\, dx = \hat{f}(k)$$

is the Fourier transform of the initial temperature. Then we have to compute the inverse Fourier transform

$$u(x,t) = \int_{-\infty}^{\infty} e^{-2\pi^2k^2t}e^{2\pi ikx}\hat{f}(k)\, dk$$

to find the solution of the heat equation. We write u in the form

$$u(x,t) = \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} e^{-2\pi^2 k^2 t} e^{2\pi i k(x-y)}\, dk\, dy$$

by switching the limits of integration. This is permissible if the absolute value (f) is integrable because then both integrals converge uniformly. Then we just have to compute the inverse Fourier transform

$$g(x) = \int_{-\infty}^{\infty} e^{-2\pi^2 k^2 t} e^{2\pi i k x}\, dk$$

to obtain our solution of the heat equations. This is accomplished by a trick

$$\begin{aligned} \frac{d}{dx} g(x) &= 2\pi i \int_{-\infty}^{\infty} k e^{-2\pi^2 k^2 t} e^{2\pi i k x}\, dk \\ &= -\frac{i}{2\pi t} e^{-2\pi^2 k^2 t} e^{2\pi i k x} |_{-\infty}^{\infty} + \frac{i}{2\pi t} \int_{-\infty}^{\infty} e^{-2\pi^2 k^2 t} \frac{\partial}{\partial k} e^{2\pi i k x}\, dk, \end{aligned}$$

by integration by parts. That is, $k e^{-2\pi^2 k^2 t} = \frac{-1}{4\pi^2 t} \frac{\partial}{\partial k} e^{-2\pi^2 k^2 t}$, and then we move the derivative onto $e^{2\pi i k x}$. The boundary terms at $k = \pm\infty$ vanish and we get

$$\frac{d}{dx} g = \frac{-x}{t} \int_{-\infty}^{\infty} e^{-2\pi^2 k^2 t} e^{2\pi i k x}\, dx = -\frac{x}{t} g(x).$$

This is a first order ordinary differential equation for $g(x)$ and its solution

$$g(x) = g(0) e^{-x^2/2t}$$

is easily obtained by integration. Now the initial condition for the ODE is

$$g(0) = \int_{-\infty}^{\infty} e^{-2\pi^2 k^2 t}\, dk = \frac{1}{\sqrt{2\pi t}}$$

by Exercise 2.1. Thus our solution becomes

$$u(x,t) = \int_{-\infty}^{\infty} f(y) \frac{e^{-(x-y)^2/2t}}{\sqrt{2\pi t}}\, dy. \tag{3.1}$$

The kernel in the integral

$$K(x,t) = \frac{\exp(x^2/2t)}{\sqrt{2\pi t}}$$

is called the heat-kernel. It is itself a solution of the heat equation for $t > 0$ and we will study its properties in the next section. The heat-kernel behaves as the Dirac delta function in Section 2.3, with a singularity at x, when $t \to 0$. Namely,

$$\begin{aligned} & \lim_{t\to 0} \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/2t}}{\sqrt{2\pi t}} f(y)\,dy \\ = & \lim_{t\to 0} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} f(x+\sqrt{2t}z)\,dz \qquad (3.2) \\ = & f(x)\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2}\,dz \\ = & f(x), \end{aligned}$$

by Exercise 2.1. We can take the limit into the integral because it converges uniformly for an integrable function f and in the first step we simply made the change of variables $z = (y-x)/\sqrt{2t}$. Now in Section 2.3 we saw that

$$\int_{-\infty}^{\infty} f(y)\delta(y-x)\,dy = f(x),$$

so

$$\lim_{t\to 0} \frac{e^{-(x-y)^2/2t}}{\sqrt{2\pi t}} = \delta(y-x).$$

On the other hand the heat-kernel decays exponentially as t increases and that means that the temperature $u(x,t)$ also decays. Of course this agrees with our intuition about heat, that it spreads and temperature quickly decreases if we turn off the heat-source.

Exercises 3.2.2

3.1 Compute the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Hint: Let the integral be equal to I, consider I^2 and use polar coordinates.

3.2 Find the heat kernel for the half-line, $x \geq 0$, and use it to solve the IVP:

$$\begin{aligned} u_t &= \frac{1}{2}u_{xx}, \qquad x \in \mathbb{R}^+,\ t \in \mathbb{R}^+ \\ u(x,0) &= f(x), \qquad x \in \mathbb{R}^+. \end{aligned}$$

This is an example of Kelvin's method of images, see Section 5.1.

3.3 Classification of PDEs and Properties of Heat Diffusion

We saw in our discussion of the wave equation that waves propagate with finite speed $c < \infty$ along characteristics and it raises the question: How fast does heat propagate? This is really the question, what are the characteristics of the heat equation? We will now discuss classification of second order linear PDEs to find the characteristics of all such equations.

A second order PDE in two variables has the form

$$Au_{x_1x_1} + Bu_{x_1x_2} + Cu_{x_2x_2} + \text{lower order terms} = 0.$$

The characteristics of this equation satisfy the equation

$$A\left(\frac{dx_2}{dx_1}\right)^2 + B\frac{dx_2}{dx_1} + C = 0.$$

It has the solution

$$\frac{dx_2}{dx_1} = -\frac{B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

if $A \neq 0$. If $B^2 - 4AC > 0$, so there exist two real solutions, the equation is called *hyperbolic*, if $B^2 = 4AC$, there is only one real solution and the equation is called *parabolic*. If no real solutions exist $B^2 - 4AC < 0$, we say that the solution is *elliptic*.

Example 3.3.1

The wave equation is hyperbolic,

$$u_{tt} - c^2 u_{xx} = 0,$$
$$A = 1, \quad B = 0, \quad C = -c^2,$$
$$B^2 - 4AC = 4c^2 > 0,$$

so

$$\frac{dx}{dt} = \pm c,$$

and we get two real solutions

$$x \pm ct = \text{constant}.$$

This was the result we obtained in Section 1.3. The heat equation is parabolic

$$u_t - \tfrac{1}{2} u_{xx} = 0,$$
$$A = 0, \quad B = 0, \quad C = -\tfrac{1}{2}$$
$$-\tfrac{1}{2}\left(\tfrac{dt}{dx}\right)^2 = 0,$$

so the characteristics are,

$$t = \text{constant}.$$

The potential equation is elliptic,

$$u_{xx} + u_{yy} = 0,$$
$$A = 1, \quad B = 0, \quad C = 1,$$
$$B^2 - 4AC = -4 < 0.$$

There are no real characteristics.

The second example shows that the characteristics for the heat equation are the horizontal lines in the $x-t$ plane,

$$t = \text{constant}.$$

This means that one cannot think of temperature at a point (x,t) as given by values from points, or intervals, on the initial line. (This is how we thought about solutions to the wave equation in Section 1.3.) In fact the next example shows that heat propagates with infinite speed.

Example 2.2 Consider a volcano where the magma (molten rock) is propagating up a narrow vertical fissure, see Figure 3.1. Suppose the eruption has just taken place and we want to compute the temperature distribution in a thin horizontal layer perpendicular to the fissure, see Figure 3.1. The initial value problem is:

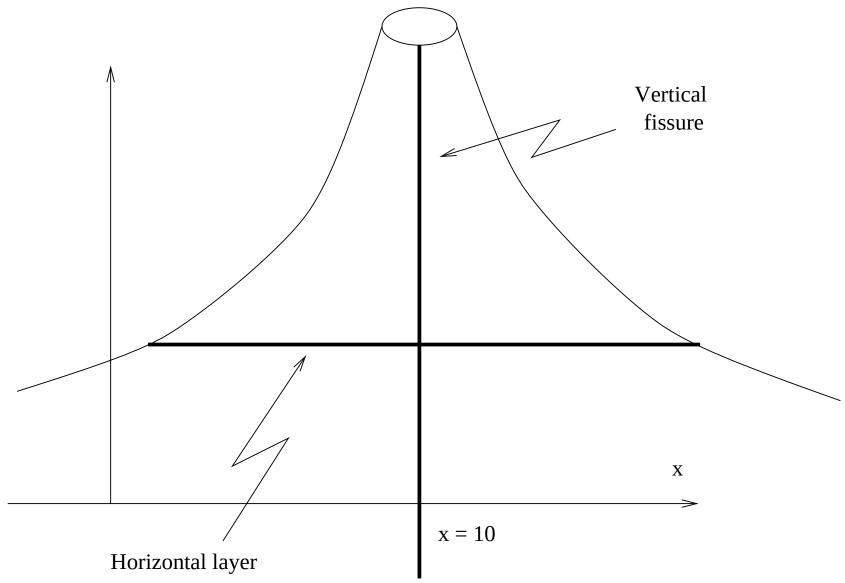

Figure 3.1: The heat in a horizontal rock layer.

$$u_t = \frac{1}{2}u_{xx}, \quad x \in \mathbb{R},\ t \in \mathbb{R}^+,$$

IC,

$$u(x,0) = T\delta(x-10).$$

We have scaled out the conductivity of the rock layer, see Remark 3.2.1, and set the temperature of the magma equal to T.

The solution of the IVP is,

$$\begin{aligned} u(x,t) &= T\int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/2t}}{(2\pi t)^{1/2}}\delta(y-10)\,dy \\ &= T\frac{e^{-(x-10)^2/2t}}{\sqrt{2\pi t}}, \end{aligned}$$

by the properties of the δ function in Section 2.3. Now let us study this solution. The first thing that we notice is that we can differentiate this solution as often as we like even though the initial data lacked one derivative. This is the first property that we note:

Property 3.3.2 *The heat equation smooths singularities.*

The next thing that we notice is that although the initial data was equal to zero for all x except $x = 10$, the above solution is greater than zero for all x, at any fixed $t > 0$. This gives us,

Property 3.3.3 *The heat equation has infinite propagation speed.*

We also see that the temperature is positive everywhere, if it was initially positive and on Figure 3.2 we illustrate the decay of the heat kernel. The heat-kernel is a Gaussian, like the normal distribution in statistics. It has the half-width $2t$ and maximum $1/\sqrt{2\pi t}$. Thus the peak decreases and the width increases with t. This gives us Property 3.3.4.

Property 3.3.4 *The heat equation smooth out irregularities, spreads the data and the solution decays to 0 as $t \to \infty$.*

Finally we consider what happens if we try to solve the heat equation backwards. Consider the IVP,

$$u_t = -\frac{1}{2}u_{xx}, \quad x \in \mathbb{R},\ t \in \mathbb{R}^+,$$

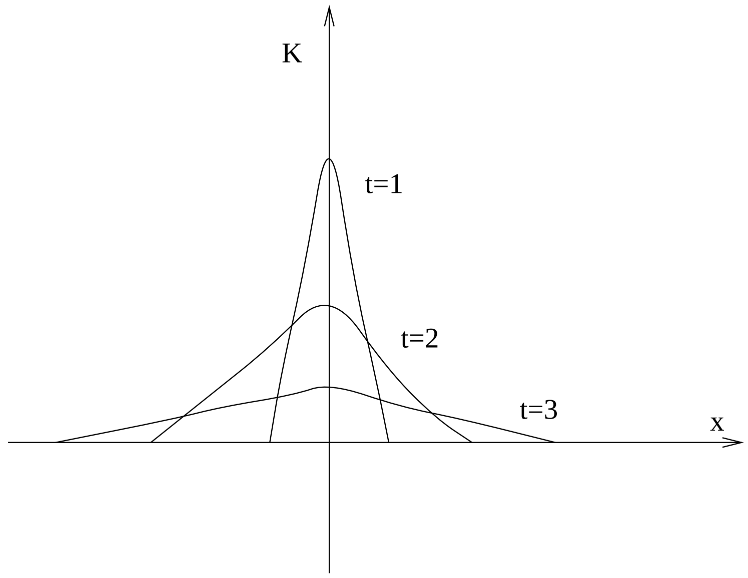

Figure 3.2: The decay of the heat kernal.

IC,

$$u(x, t_o) = f(x).$$

By symmetry in t, the solution must be

$$u(x,t) = \int_{-\infty}^{\infty} \frac{e^{(x-y)^2/2(t-t_o)}}{\sqrt{2\pi(t-t_o)}}\, f(y)\, dy.$$

But this integral *diverges* for all values of x (unless f decays faster than $e^{-x^2/2(t-t_o)} \to 0$) and the factor $1/\sqrt{2\pi(t-t_o)}$ blows up, as $t \to t_o$. Thus we get,

Property 3.3.5 *The solution of the heat equation blows up everywhere for $t < 0$.*

The Properties 3.3.2–3.3.5 are clearly very different from Properties 1.4.2–1.4.4 of the wave equation. The former are typical properties of solutions to parabolic PDEs.

Exercises 3.3.6

3.3 Solve the IVP,

$$u_t = \frac{1}{2}u_{xx}, \quad x \in \mathbb{R},\ t \in \mathbb{R},$$

IC,

$$u(x,0) = 3x.$$

3.4 Pose and solve the IVP for the temperature in a semi-infinite rod when the finite end (at $x = 0$) is insulated. The initial temperature distribution is $e^{-(x-2)^2}$. Show that the heat decreases with time and find the limit

$$\lim_{t\to\infty} u(3,t),$$

where $u(x,t)$ is the temperature.

3.4 A Heat Source and The Well Posedness of the Heat Equation

We now find a solution of the heat equation including a heat source that is warming up every part of the rod. The IVP is,

$$u_t = \frac{1}{2}u_{xx} + F(x,t), \quad x \in \mathbb{R},\ t \in \mathbb{R}^+,$$

IC,

$$u(x,0) = f(x).$$

$F(x,t)$ is the function that models the heat source. We will use the Fourier transform to solve this IVP but first we use the superposition principle 2.1.2 to split the problem into a homogeneous IVP,

$$\begin{aligned} u_t &= \frac{1}{2}u_{xx}, \quad x \in \mathbb{R},\ t \in \mathbb{R}^+, \\ u(x,0) &= f(x), \end{aligned}$$

and the inhomogeneous IVP with initial data zero,

$$\begin{aligned} u_t &= \frac{1}{2}u_{xx} + F(x,t), \quad x \in \mathbb{R},\ t \in \mathbb{R}^+, \\ u(x,0) &= 0. \end{aligned}$$

The solution of the original problem is then the sum of the solutions of the latter two problems by the superposition principle 2.1.2.

The Fourier transform of the inhomogeneous equation is,

$$\hat{u}_t = -2\pi^2 k^2 \hat{u} + \hat{F}(k,t).$$

This is a first order linear ODE. We solve it by finding an integration factor $e^{2\pi^2 k^2 t}$, then

$$\frac{d}{dt}(e^{2\pi^2 k^2 t}\hat{u}) = e^{2\pi^2 k^2 t}\hat{F}(k,t)$$

and

$$\hat{u}(k,t) = \int_0^t e^{-2\pi^2 k^2 (t-s)}\hat{F}(k,s)\,ds,$$

since the initial condition vanishes,

$$\hat{u}(k,0) = \int_{-\infty}^{\infty} u(x,0)e^{-2\pi i k x}\,dx = 0.$$

Then we apply the inverse Fourier transform and compute the heat kernel as in Section 3.2,

$$\begin{aligned} u(x,t) &= \int_0^t \int_{-\infty}^{\infty} e^{-2\pi^2 k^2 (t-s)} e^{2\pi i k x}\hat{F}(k,s)\,dk\,ds \\ &= \int_0^t \int_{-\infty}^{\infty} F(y,s)\frac{e^{-(x-y)^2/2(t-s)}}{(2\pi(t-s))^{1/2}}\,dy\,ds. \end{aligned}$$

Finally, the solution of the original IVP is the sum of the above function and the solution of the homogeneous IVP 3.1, from Section 3.2,

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \frac{f(y)e^{-(x-y)^2/2t}}{\sqrt{2\pi t}}\,dy \\ &+ \int_0^t \int_{-\infty}^{\infty} \frac{F(y,s)e^{-(x-y)^2/2(t-s)}}{(2\pi(t-s))^{1/2}}\,dy\,ds. \end{aligned} \tag{3.3}$$

We can now state the conditions on f and F that suffice to give us a solution of the IVP for the inhomogeneous heat equation.

Theorem 3.4.1 *If* $f \in C^0(\mathbb{R})$, $F \in C^\infty(\mathbb{R} \times \mathbb{R}^+)$ *and both are summable*

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty, \quad \int_0^\infty \int_{-\infty}^{\infty} |F(x,t)|\, dx\, dt < \infty,$$

then

$$u(x,t) = \int_{-\infty}^{\infty} \frac{f(y) e^{-(x-y)^2/2t}}{\sqrt{2\pi t}}\, dy + \int_0^t \int_{-\infty}^{\infty} \frac{F(y,s) e^{-(x-y)^2/2(t-s)}}{(2\pi(t-s))^{1/2}}\, dy\, ds,$$

is the unique solution of the IVP,

$$u_t = \frac{1}{2} u_{xx} + F(x,t), \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+,$$

IC,

$$u(x,0) = f(x), \quad x \in \mathbb{R}.$$

Moreover, $u(x,t) \in C^\infty(\mathbb{R} \times \mathbb{R}^+)$, *for* $t > 0$ *and*

$$\lim_{t \to 0} \sup_{x \in \mathbb{R}} |u(x,t) - f(x)| = 0.$$

Proof: We write the solution in the form

$$u(x,t) = \int_{-\infty}^{\infty} K(x-y,t) f(y) dy + \int_0^t \int_{-\infty}^{\infty} K(x-y,t-s) F(y,s) dy ds$$

where $K(x,t)$ is the heat kernal from Section 3.2. Recall from Section 3.2 that as $t \to 0$, $K \to \delta$. This show that

$$\lim_{t \to 0} u(x,t) = \int_{-\infty}^{\infty} \delta(x-y) f(y) dy = f(x)$$

because

$$\lim_{t \to 0} \int_0^t \int_{-\infty}^{\infty} K(x-y,t-s) F(y,s) dy ds = \lim_{t \to 0} \int_0^t F(x,s) ds = 0.$$

This means that $u(x,t)$ assumes its initial data as $t \to 0$. u is smooth because the exponential decay of K lets us differentiate under the ingral sign as often as we like

$$\int_{-\infty}^{\infty} (\partial_x^n K(x-y,t)) f(y) dy \ \in\ C^0(\mathbb{R} \times \mathbb{R}^+),$$

and

$$\int_0^t \int_{-\infty}^{\infty} (\partial_x^n K(x-y,t-s))F(y,s)dyds =$$
$$\int_0^t \int_{-\infty}^{\infty} K(x-y,t-s)(\partial_y^n F(y,s))dyds \in C^0(\mathbb{R} \times \mathbb{R}^+),$$

by integration by parts, the t and mixed derivatives are similar. Now we differentiate the solution with respect to t to get

$$\begin{aligned} \partial_t \int_{-\infty}^{\infty} K(x-y,t)f(y)dy &+ \int_{-\infty}^{\infty} \delta(x-y)F(y,t)dy \\ &+ \int_0^t \int_{-\infty}^{\infty} \partial_t K(x-y,t-s)F(y,s)dyds \end{aligned}$$

and

$$\begin{aligned} (\partial_t - \frac{1}{2}\partial_{xx})u &= \int_{-\infty}^{\infty} (\partial_t - \frac{1}{2}\partial_{xx})K(x-y,t))f(y)dy \\ &+ \int_{-\infty}^{\infty} \delta(x-y)F(y,t)dy \\ + \int_0^t \int_{-\infty}^{\infty} (\partial_t &- \frac{1}{2}\partial_{xx})K(x-y,t-s)F(y,s)dyds \\ &= F, \end{aligned}$$

because

$$(\partial_t - \frac{1}{2}\partial_{xx})K(x,t) = 0.$$

This shows that u satisfies the inhomogeneous heat equation. The uniform limit is evaluated from

$$\lim_{t\to 0} |u(x,t) - f(x)| \leq$$
$$\lim_{t\to 0} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} |f(x+\sqrt{2t}z) - f(x)|\, dz$$
$$= 0,$$

by the formula 3.2 in Section 3.2 and Exercise 3.1. The exponential decay implies that the integral converges uniformely and we can take the integral inside. Finally the uniqueness of the solution follows from the derivation of the solution in Section 3.2. All the steps in the derivation are justified, for $t > 0$, with the integrability conditions on f and F and this gives a unique solution.

□

3.5 The Heat Equation on a Finite Interval

The heat equation for a finite rod can be solved, as the wave equation in Section 2.2, by extending the initial data to the whole line, in accordance with the boundary conditions. If the ends of the rod are kept *cold*, we have Dirichlet (or fixed) BC,

$$u(0,t) = 0 = u(\ell,t), \quad t \in \mathbb{R}^+,$$

assuming that the rod has length ℓ. If the ends of the rod are *insulated* we get Neumann (or free) BC,

$$u_x(0,t) = 0 = u_x(\ell,t), \quad t \in \mathbb{R}^+.$$

The initial data is extended as an *odd function*, with period 2ℓ, with Dirichlet boundary condition, see Figure 2.3. It is extended as an *even function*, with period 2ℓ, see Figure 2.4, with Neumann boundary conditions.

Example 3.5.1

Solve the IVP,

$$u_t = \frac{1}{2} u_{xx}, \quad 0 \le x \le \ell, \; t \in \mathbb{R}^+$$

IC,

$$u(x,0) = 1, \quad 0 \le x \le \ell,$$

BC,

$$u_x(0,t) = 0 = u_x(\ell,t), \quad t \in \mathbb{R}^+.$$

Solution: We extend the initial data as an even function because we have Neumann boundary conditions, see Figure 3.3. The solution is

$$u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/2t}}{\sqrt{2\pi t}} \, dy$$

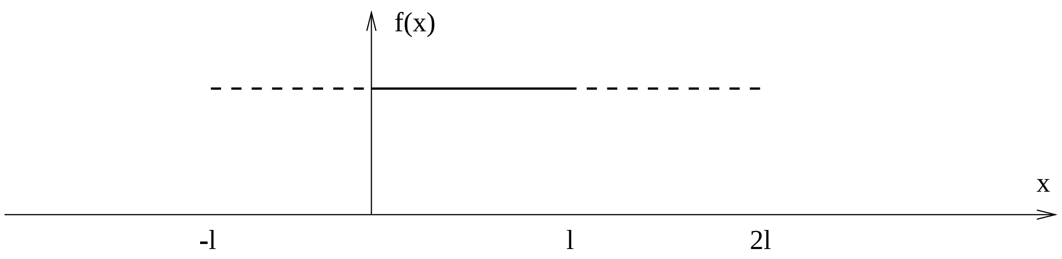

Figure 3.3: The even extension of f.

by the formula in Section 3.2. We make a change of variables

$$z = \frac{y-x}{\sqrt{2t}}, \quad dz = \frac{dy}{\sqrt{2t}},$$

then

$$u(x,t) = \frac{\sqrt{2t}}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-z^2}\, dz = 1,$$

because

$$\int_{-\infty}^{\infty} e^{-z^2}\, dz = \sqrt{\pi},$$

by Exercise 3.1. This means that the solution is

$$u(x,t) \equiv 1,$$

see Figure 3.4. A moment's thought tells you without any computations that this is exactly what the solution must be. Why?

Exercises 3.5.2

3.5 Find the temperature distribution in a rod of length ℓ, if initially $u(x,0) = cx$. One of the ends (at $x = 0$) of the rod is kept cold, the other one (at $x = \ell$) is insulated.

3.6 Solve the IVP,

$$u_t = \frac{1}{2} u_{xx}, \quad 0 \leq x \leq \ell,\ t \in \mathbb{R}^+,$$

IC,

$$u(x,0) = f(x),$$

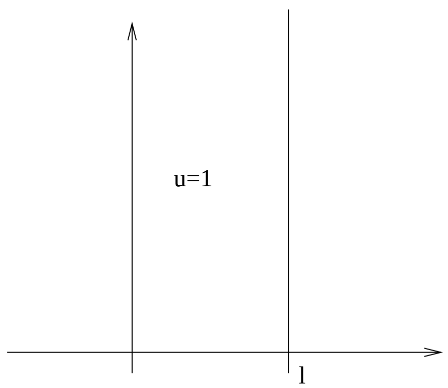

Figure 3.4: The heat in the rod.

BC,

$$u(0,t) = a, \quad u(\ell,t) = b, \ t \in \mathbb{R}^+.$$

Hint: Let $u = a + (b-a)x/\ell + v$ and solve for v.

3.6 The Schrödinger Equation

The Schrödinger equation

$$\begin{aligned} i\hbar \tfrac{\partial \psi}{\partial t} &= -\tfrac{\hbar^2}{2m}\Delta\psi, \qquad x \in \mathbb{R}, \ t \in \mathbb{R}^+, \\ \psi(x,0) &= f(x), \qquad\qquad x \in \mathbb{R}, \end{aligned}$$

is the fundamental equation in quantum mechanics. We can solve it in exactly the same manner as we solved the heat equation in Section 3.2 and get the solution

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{f(y)e^{i|x-y|^2/2t}}{(2\pi i t)^{1/2}}\, dy,$$

if we set $\hbar = 1 = m$. This is the wave function of the free particle moving on an infinite line. Notice that the formula amounts to making the change of variables,

$$t \to it,$$

in the solution of the heat equation. The similarity ends there because the properties of ψ turn out to be more similar to those of the solutions of the wave equation, in Section 1.4, than the properties of the solutions of the heat equation. If $\psi(x,t)$ solves the Schrödinger's equation with $\hbar/m = 1$, the $\psi(x, \hbar/mt)$ solves the original problem by the Remark 3.2.1. Then another application of the superposition principle 2.1.2 gives the solution to the inhomogeneous Schrödinger's equation,

$$\begin{aligned} i\hbar\tfrac{\partial\psi}{\partial t} &= -\tfrac{\hbar^2}{2m}\Delta\psi + F(x,t), && x \in \mathbb{R},\ t \in \mathbb{R}^+, \\ \psi(x,0) &= f(x), && x \in \mathbb{R}\ . \end{aligned}$$

Namely,

$$\begin{aligned} \psi(x,t) &= \int_{-\infty}^{\infty} \frac{f(y)e^{i\hbar|x-y|^2/2mt}}{(2\pi i\hbar t/m)^{1/2}}\,dy \\ &+ \int_0^t \int_{-\infty}^{\infty} \frac{F(y,s)e^{i\hbar|x-y|^2/2m(t-s)}}{(2\pi i\hbar(t-s)/m)^{1/2}}\,ds\,dy. \end{aligned} \tag{3.4}$$

Chapter 4

The Potential Equation

4.1 The Dirichlet and the Neumann Problem

If the temperature distribution in some three-dimensional body is not a function of time, for example if the surface temperature is constant in time, then the temperature distribution obeys the potential equation,

$$\Delta u = 0,$$

where the operator Δ is called the Laplacian. In three dimensions,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

In two dimensions, the solution of the potential equation

$$u_{xx} + u_{yy} = 0,$$

describes for example the equilibrium shape of the surface of a drum when the shape of the boundary is given, and the fluid potential in an incompressible and irrotational fluid. In addition, the potential equation determines an electric potential around a distribution of electric charges, etc. All of these examples have in common that they do not involve propagation or diffusion. This distinguishes them from wave and heat equations. They do not involve dynamics and evolution of initial data and consequently the correct problem to pose for them is a boundary value problem (BVP). Two types of boundary conditions (BC) are most common.

The Dirichlet BVP:

$$\Delta u = 0, \qquad x \in \mathcal{D} \subset \mathbb{R}^3$$

BC,

$$u(x) = g(x), \qquad x \in \partial\mathcal{D}.$$

$\mathcal{D}$ is an open region in $\mathbb{R}^3$, $\partial\mathcal{D}$ is the boundary of $\mathcal{D}$ and the BC gives the value of $u(x)$ on the boundary, see Figure 4.1. The other most

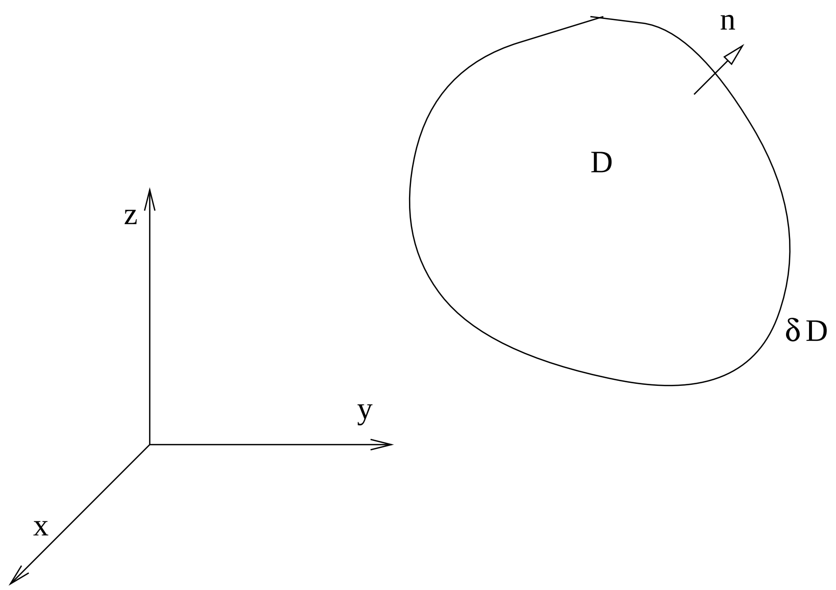

Figure 4.1: A region and its boundary.

common BC is the Neumann BC.

The Neumann BVP:

$$\begin{aligned} \Delta u &= 0, \qquad x \in \mathcal{D}, \\ \frac{\partial u}{\partial n} &= f(x), \qquad x \in \partial\mathcal{D}. \end{aligned}$$

Here $\frac{\partial u}{\partial n} = n \cdot \nabla u$ is the directional derivative in the direction of the normal n out of $\mathcal{D}$. We can think about this boundary condition as specifying the flux through the surface. Sometimes the Dirichlet condition is given for a portion of the boundary, say $\partial\mathcal{D}_1$, and the Neumann condition for the rest $\partial\mathcal{D}_2$, $\partial\mathcal{D} = \partial\mathcal{D}_1 \cup \partial\mathcal{D}_2$. Occasionally the two are mixed in a Robin BC:

$$au + b\frac{\partial u}{\partial n} = g(x),\ \ x \in \partial\mathcal{D},$$

a, b constants. We have posed these problems in three dimensions but they can be posed in any dimension $n \geq 1$, by simply letting $\mathcal{D} \subset \mathbb{R}^n$.

Solutions of the potential equation are called *harmonic functions*.

Example 4.1.1

Show that $1/|\mathbf{x}|,\ \mathbf{x} \neq 0$, is a solution of the potential equation in $\mathbb{R}^3$.

Solution: We differentiate

$$\frac{\partial}{\partial x}\frac{1}{(x^2+y^2+z^2)^{1/2}} = \frac{-x}{(x^2+y^2+z^2)^{3/2}},$$

and

$$\frac{\partial^2}{\partial x^2}\frac{1}{|\mathbf{x}|} = \frac{-1}{(x^2+y^2+z^2)^{3/2}} + \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}.$$

This implies that

$$\begin{aligned}\Delta\frac{1}{|\mathbf{x}|} &= \frac{-3}{(x^2+y^2+z^2)^{3/2}} + \frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} \\ &= 3\left(\frac{-1}{|\mathbf{x}|^3} + \frac{1}{|\mathbf{x}|^3}\right) = 0. \quad \square\end{aligned}$$

Exercises 4.1.2

4.1 Show that $u = \log|\mathbf{x}|$, where $|\mathbf{x}| = (x^2 + y^2)^{1/2}$, is a solution of

$$\Delta u = 0, \qquad \mathbf{x} \neq 0,$$

$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, is the two-dimensional Laplacian and $u = |\mathbf{x}|^{n-2}$, where $|\mathbf{x}| = (x_1^2 + \ldots + x_n^2)^{1/2}$ is a solution of the n-dimensional potential equation,

$$\Delta u = 0, \qquad \mathbf{x} \neq \mathbf{0},$$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}.$$

4.2 The Divergence Theorem

We need several mathematical tools to solve the potential equation, the main tool being the divergence theorem from vector calculus.

If $\mathcal{D}$ is a set then $(x, y, z)_0$ is an *interior point* of $\mathcal{D}$ if it is contained in a ball of radius r,

$$\mathcal{B} = \{(x, y, z) \mid |(x, y, z) - (x, y, z)_0| < r\} \subset \mathcal{D},$$

inside $\mathcal{D}$.

The *distance* between two points is

$$|(x, y, z) - (x, y, z)_0| = ((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2)^{1/2}.$$

A point $(x, y, z)_0$ is *exterior* to $\mathcal{D}$ if there exists a ball $\mathcal{B}$ of radius r around $(x, y, z)_0$ such that

$$\mathcal{D} \cap \mathcal{B} = \emptyset,$$

or the intersection of $\mathcal{D}$ and $\mathcal{B}$ is empty.

If all balls around $(x, y, z)_0$ contain both points interior and exterior to $\mathcal{D}$, then $(x, y, z)_0$ is a *boundary point* of $\mathcal{D}$.

$\mathcal{D}$ is an open set if and only if all its points are interior points. The *complement* of $\mathcal{D}$,

$$\mathbb{R}^3 \backslash \mathcal{D},$$

is the set of all points exterior to $\mathcal{D}$. $\mathcal{D}$ is a *closed set* if and only if it contains all its boundary point. This is equivalent to the conditon that the complement of $\mathcal{D}$ is open.

We let $\partial\mathcal{D}$ denote the boundary of $\mathcal{D}$. Then we can close $\mathcal{D}$;

$$\overline{\mathcal{D}} = \mathcal{D} \cup \partial\mathcal{D},$$

is called the closure of $\mathcal{D}$. If $\mathcal{D}$ is closed then

$$\overline{\mathcal{D}} = \mathcal{D}.$$

A set is *connected* if one can connect every two points in $\mathcal{D}$ by a curve all of whose points lie in $\mathcal{D}$. An open connected set is called *a region.*

A region $\mathcal{D}$ is bounded if there exists a ball $\mathcal{B}$ such that

$$\mathcal{D} \subset \mathcal{B}.$$

An unbounded region $\mathcal{D}$ contains a neighborhood of infinity, if there exists a ball $\mathcal{B}$, such that all points outside $\mathcal{B}$ lie in $\mathcal{D}$,

$$\mathbb{R}^3 \backslash \mathcal{B} \subset \mathcal{D}.$$

A region is *simply-connected* if one can contract all closed balls lying in the region (loops in two dimensions) to a point without ever leaving the region.

These concepts allow us to work with the divergence theorem and other technical tools.

Theorem 4.2.1 (The Divergence Theorem) *If* $\mathbf{v}(\mathbf{x}) \in [C^1(\mathcal{D})]^3$ *is a vector-valued function such that* $\mathbf{v}(\mathbf{x}) \in [C^0(\partial\mathcal{D})]^3$, *then*

$$\int_{\mathcal{D}} \nabla \cdot \mathbf{v}\, dx = \int_{\partial\mathcal{D}} \mathbf{n} \cdot \mathbf{v}\, d\sigma,$$

n being the outward normal of $\partial\mathcal{D}$ and $d\sigma$ the surface element.

The divergence theorem is proven by integration by parts. A nice proof can be found in Marsden and Weinstein, Calculus Vol. III, [7].

Corollary 4.2.2 *If the vector* $\mathbf{v} \in [C^1(\mathcal{D})]^3 \cap [C^0(\partial\mathcal{D})]^3$*, and the scalars* $u, w \in C^1(\mathcal{D}) \cap C^0(\partial\mathcal{D})$*, then the Green's identities hold,*

$$\int_{\mathcal{D}} \mathbf{v} \cdot \nabla u \, dx = \int_{\partial\mathcal{D}} (\mathbf{n} \cdot \mathbf{v}) u \, d\sigma - \int_{\mathcal{D}} u \nabla \cdot \mathbf{v} \, dx, \tag{4.1}$$

and if $w \in C^2(\mathcal{D}) \cap C^1(\partial\mathcal{D})$*,*

$$\int_{\mathcal{D}} \nabla u \cdot \nabla w \, dx = \int_{\partial\mathcal{D}} u \frac{\partial w}{\partial n} \, d\sigma - \int_{\mathcal{D}} u \Delta w \, dx, \tag{4.2}$$

and if $u, w \in C^2(\mathcal{D}) \cap C^1(\partial\mathcal{D})$*,*

$$\int_{\mathcal{D}} (u\Delta w - w\Delta u) \, dx = \int_{\partial\mathcal{D}} \left(u \frac{\partial w}{\partial n} - w \frac{\partial u}{\partial n} \right) d\sigma. \tag{4.3}$$

Proof of 4.1 and 4.3: We let the vector in the divergence theorem be $h = u\mathbf{v}$, then

$$\int_{\mathcal{D}} \nabla \cdot (u\mathbf{v}) \, dx = \int_{\partial\mathcal{D}} u\mathbf{n} \cdot \mathbf{v} \, d\sigma$$

but

$$\nabla \cdot (u\mathbf{v}) = (\nabla u) \cdot \mathbf{v} + u \nabla \cdot \mathbf{v},$$

thus

$$\int_{\mathcal{D}} \mathbf{v} \cdot \nabla u \, dx = \int_{\partial\mathcal{D}} (\mathbf{n} \cdot \mathbf{v}) u \, d\sigma - \int_{\mathcal{D}} u \nabla \cdot \mathbf{v} \, dx.$$

We obtain 4.3 by interchanging u and w in 4.2 and taking the difference.
□

Exercises 4.2.3

4.2 Prove the Green's identity 4.2.

4.3 The Electric Potential

Coulomb's law says that the electric field at the distance $|\mathbf{x}|$ from an electric charge q is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0|\mathbf{x}|^2} \cdot \frac{\mathbf{x}}{|\mathbf{x}|},$$

where ϵ_0 is the (vacuum) permeability. Gausses law then states that the electric flux through the surface of a region $\mathcal{D}$ is

$$F = \int_{\partial\mathcal{D}} \mathbf{E}\cdot\mathbf{n}\, d\sigma = \int_{\mathcal{D}} \frac{\zeta(\mathbf{x})}{\epsilon_0}\, dx,$$

where $\zeta(\mathbf{x})$ is the charge distribution in $\mathcal{D}$. If we apply the divergence theorem 4.2.1, we get

$$\int_{\partial\mathcal{D}} \mathbf{E}\cdot\mathbf{n}\, d\sigma = \int_{\mathcal{D}} \nabla\cdot\mathbf{E}\, dx$$

and comparison with the integral above gives

$$\nabla\cdot\mathbf{E} = \frac{\zeta(\mathbf{x})}{\epsilon_0}.$$

If the region $\mathcal{D}$ does not contain any charges then the divergence of the field vanishes

$$\nabla\cdot\mathbf{E} = 0.$$

The electric potential is the integral of the electric field along some trajectory,

$$V(\mathbf{x}) = -\int_\infty^{\mathbf{x}} \mathbf{E}\cdot\mathbf{d}\ell,$$

where $d\ell$ denotes the line element. We assume that the potential is zero at infinity and that the integral is convergent. Potential theory deals with electric fields which are independent of time, $\nabla\times\mathbf{E} = 0$, then the integral above does not depend on the path and V is a well-defined function

$$V(\mathbf{x}) = \int_\infty^{\mathbf{x}} \nabla V\cdot d\ell.$$

This implies that

$$\mathbf{E} = -\nabla V.$$

If $\mathcal{D}$ contains no charge, Gausses law says that

$$\nabla \cdot \mathbf{E} = 0$$

or

$$\Delta V = \nabla \cdot \nabla V = 0$$

which is the potential equation. If $\mathcal{D}$ contains the charge distribution ζ we obtain Poisson's equation

$$\Delta V = -\frac{\zeta(\mathbf{x})}{\epsilon_{\mathbf{0}}}.$$

If the charge distribution is simply a point charge q located at the origin then

$$\zeta(\mathbf{x}) = \frac{q}{4\pi}\delta(\mathbf{x}),$$

where δ is the Dirac δ-function from Section 2.3. We will now show how to solve the PDE

$$\Delta V = \frac{q}{4\pi}\delta(\mathbf{x})$$

using a physical law.

Suppose we want to find the electric potential at a distance $|\mathbf{x}|$ from the charge q. Then we can simply integrate the electric field given by the Coulomb's law above, namely,

$$\begin{aligned}
\mathbf{E} &= \frac{q}{4\pi\epsilon_0|\mathbf{x}|^{\mathbf{2}}} \cdot \frac{\mathbf{x}}{|\mathbf{x}|}, \\
V(\mathbf{x}) &= -\frac{q}{4\pi\epsilon_o}\int_{\infty}^{\mathbf{x}} \frac{1}{|\mathbf{x}|^{\mathbf{2}}}\frac{\mathbf{x}}{|\mathbf{x}|} \cdot d\mathbf{x} \\
&= -\frac{q}{4\pi\epsilon_0}\int_{\infty}^{|\mathbf{x}|} \frac{1}{|\mathbf{x}|^2} d|\mathbf{x}| = \frac{q}{4\pi|\mathbf{x}|},
\end{aligned}$$

as in Example 4.1.1, because $\frac{d|\mathbf{x}|}{ds} = \frac{d}{ds}(\mathbf{x}\cdot\mathbf{x})^{1/2} = \frac{\mathbf{x}\cdot d\mathbf{x}/ds}{(\mathbf{x}\cdot\mathbf{x})^{1/2}}$, s being some parameter. Thus

$$V(\mathbf{x}) = \frac{q}{4\pi\epsilon_0|\mathbf{x}|},$$

is the desired solution of Poisson's equation. This example shows that one can sometimes use physical laws to solve PDEs.

4.4 The Properties of Harmonic Functions

We can find many of the properties of the solutions of the potential equation without solving the equation if we use the Green's identities in Corollary 4.2.2. We simply let $w = u$ in 4.2 and obtain

$$\int_{\Omega} (\nabla u)^2 \, dx = \int_{\partial\Omega} u \frac{\partial u}{\partial n} \, d\sigma, \tag{4.4}$$

because $\Delta u = 0$. This identity gives us a lot of information. First if $u = 0$ or $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ then $\nabla u = 0$ in Ω by 4.4. Therefore, $u = \text{constant} = 0$, if $u = 0$ on $\partial\Omega$. This gives us the first property of harmonic functions.

Theorem 4.4.1 *If a harmonic function vanishes on all the boundary of a bounded region Ω, then $u = 0$ everywhere in Ω.*

If the normal derivative $\frac{\partial u}{\partial n}$ of a harmonic function u is zero on the whole boundary of a bounded region Ω, the $u =$ constant everywhere in Ω.

Then if v and w are two solutions and we let $u = v - w$ in Theorem 4.4.1, we get the following conclusion.

Theorem 4.4.2 *The solution of the Dirichlet BVP for a bounded region is unique. The solution of the Neumann BVP on such a region is unique up to a constant.*

Next we exchange u and w and let $w = 1$ in the Green's identity 4.2. This gives us a conclusion related to Theorem 4.4.2 and consequently called a Corollary to that Theorem. Corollary 4.4.3 restricts the boundary data that we can choose for the Neumann problem.

Corollary 4.4.3 *The Neumann BVP has a solution only if*

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} \, d\sigma = \int_{\partial\Omega} f \, d\sigma = 0.$$

Warning: We have up to this point distinguished between vectors $\mathbf{x}$ and scalars x. But from now on we will let the context explain which is which. The readers will have to remind themselves frequently in the beginning that x is a vector in two or three dimensions, gradually they will get used to this and eventually find it more convenient.

The most useful property of harmonic functions is that their value at the center of a ball equals their *mean-value* over the surface of the ball. We use the Green's identity 4.2, and let $w = 1$, then

$$\int_{\partial B} \frac{\partial u}{\partial n}\, d\sigma = 0,$$

because $\Delta u = 0$. But because $\mathcal{B}$ is a sphere $\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$ and we get

$$r^2 \frac{\partial}{\partial r} \int_{\partial B} u(x + r\omega)\, d\omega = 0,$$

where

$$\omega = (\cos\theta\cos\varphi, \sin\theta\cos\varphi, \sin\varphi)$$

and

$$d\omega = \sin\varphi\, d\theta\, d\varphi.$$

This means that

$$\frac{1}{4\pi} \int_{\partial B} u(x + r\omega)\, d\omega = \text{constant},$$

but the integral is uniformly convergent so we can take the limit

$$\lim_{r\to 0} \frac{1}{4\pi} \int_{\partial B} u(x + r\omega)\, d\omega = u(x)$$

inside the integral. The argument is similar in two dimensions and we can state the result.

Theorem 4.4.4 (The Mean-Value Theorem) *Let $\mathcal{B}$ be a sphere with center x and u a harmonic function. Then*

$$\begin{aligned} u(x) &= \frac{1}{4\pi} \int_{\partial B} u(x + r\omega)\, d\omega, \quad d\sigma = r^2\, d\omega, \\ &= \frac{1}{4\pi} \int_0^{2\pi}\!\!\int_{-\pi}^{\pi} u(x + r\cos\theta\cos\varphi, y + r\sin\theta\cos\varphi, z + r\sin\varphi) \sin\varphi\, d\theta d\varphi, \end{aligned}$$

in three dimensions, and

$$\begin{aligned} u(x) &= \frac{1}{2\pi}\int_{\partial\mathcal{D}} u(x+r\omega)\,d\omega, \quad d\sigma = r\,d\omega, \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(x+r\cos\theta, y+r\sin\theta)\,d\theta, \end{aligned}$$

in two dimensions.

Remark 4.4.5 *Notice that both the integrals are independent of the radius r of the ball $\mathcal{B}$ in three dimensions and the disk $\mathcal{D}$ in two dimensions.*

A harmonic function must attain its maximum and its minimum on the boundary of a region. If u has a maximum in a point x inside a region Ω and is not a constant function, then $u(x+r\omega) < u(x)$ on some sphere with radius r and center x,

$$u(x) > \frac{1}{4\pi}\int_{\partial B} u(x+r\omega)\,d\omega = u(x),$$

by the mean-value theorem. Therefore $u(x) = u(x+r\omega)$ on all spheres with center x and we can cover Ω with spheres on which u is a constant. If $y \in \Omega$, we connect x and y by a curve lying inside Ω. We can do this because Ω is connected and cover the curve with overlapping spheres on which u is a constant. This implies that

$$u(y) = u(x) = \text{constant}.$$

This give the next property of harmonic functions.

Theorem 4.4.6 (The Maximum-Value Theorem) *If a harmonic function has a maximum (or a minimum) inside a connected region Ω, then u is a constant on Ω.*

The above statement for the minimum follows by the same argument, if we consider $-u$. The Minimum- and the Maximum-Value Theorems combine to give the Corollary.

Corollary 4.4.7 *A harmonic function must assume both its minimum and its maximum on the boundary $\partial\Omega$ of a bounded region Ω.*

We can now show that the solution to the Dirichlet BVP is stable. We let v be a solution with BC, $v = g$ on $\partial\Omega$ and w a solution with BC, $w = g + g_0$ on $\partial\Omega$, where $\max_{\partial\Omega} |g_0| \leq \epsilon$. Then if $u = v - w$

$$\max_{\partial\Omega} |u| = \max_{\partial\Omega} |g_0| \leq \epsilon.$$

This implies that

$$-\epsilon \leq u \leq \epsilon, \quad \text{on } \partial\Omega$$

and therefore by Corollary 4.4.7

$$-\epsilon \leq u \leq \epsilon, \quad \text{in } \Omega.$$

Thus

$$|v - w| \leq \epsilon, \quad \text{in } \Omega.$$

This establishes the stability property.

Theorem 4.4.8 *If Ω is a bounded region, then the solution to the Dirichlet BVP is stable.*

Exercises 4.4.9

4.3 Explain why the following statements seem to be contradictions but are not,

a. $u = 1$ and $u = 1/|x|$ are both harmonic functions in the unit ball with boundary data $u = 1$ on the boundary $|x| = 1$.

b. $u = -\log|x|$, is a harmonic function in the unit disk, with $u = 0$ on the boundary but $u > 0$ inside the unit disk.

4.5 A Fundamental Solution of the Potential Equation in $\mathbb{R}^n$.

We write the Laplace operator in polar coordinates in n dimensions,

$$r = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2} = |x|,$$
$$\tfrac{\partial}{\partial x} = \tfrac{\partial r}{\partial x}\tfrac{\partial}{\partial r} = \tfrac{x}{r}\tfrac{\partial}{\partial r}, \quad \tfrac{\partial^2}{\partial r^2} = \tfrac{1}{r}\tfrac{\partial}{\partial r} + \tfrac{x^2}{r^2}\tfrac{\partial^2}{\partial x^2} - \tfrac{x^2}{r^3}\tfrac{\partial}{\partial r}$$

so

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{(n-1)}{r}\frac{\partial}{\partial r} + \text{angular derivatives.}$$

We seek a solution that depends only on r, then

$$\begin{aligned} u_{rr} + \frac{(n-1)}{r}u_r &= 0, \\ \frac{u_{rr}}{u_r} &= -\frac{(n-1)}{r}, \\ \log u_r &= -(n-1)\log r + \log c_1, \\ u_r &= \frac{c_1}{r^{n-1}}. \end{aligned}$$

If $n = 1$, then

$$u = c_1|x| + c_2.$$

If $n = 2$, then

$$u = c_1 \log|x| + c_2.$$

If $n \geq 3$, then

$$u = \frac{c_1}{|x|^{n-2}} + c_2.$$

$1/|x|^{n-2}$ is called a fundamental solution of the potential equation in $\mathbb{R}^n$. It has a singularity at $x = 0$. We can think about it physically as the electric potential in the distance $|x|$ from a charged particle with charge q at the origin

$$V(x) = \frac{q}{4\pi\epsilon_0|x|},$$

in three dimensions, $c_2 = q/4\pi\epsilon_0$, see Section 4.3.

We now consider a bounded region Ω, see Figure 4.2, and cut out of it a sphere with center x and radius ϵ. We denote the remaining region $\Omega' = \Omega \backslash \mathcal{B}$. The Green's identity 4.3 implies that

$$\begin{aligned} 0 &= \int_{\Omega} (u\Delta(1/r) - \Delta u(1/r))\, dx \\ &= -\int_{\partial\mathcal{B}} \left[u\frac{\partial(1/r)}{\partial n} - \frac{\partial u}{\partial n}(1/r) \right] d\sigma + \int_{\partial\Omega} \left[u\frac{\partial(1/r)}{\partial n} - \frac{\partial u}{\partial n}(1/r) \right] d\sigma, \end{aligned}$$

because u is harmonic and $1/r$ is also harmonic in Ω'. The singularity of $1/r$ lies outside Ω' (inside $\mathcal{B}$), and on $\partial\mathcal{B}$ the normal is in the direction of the radius so $\frac{\partial}{\partial n} = \frac{-\partial}{\partial r}$ on $\partial\mathcal{B}$. The last equation becomes

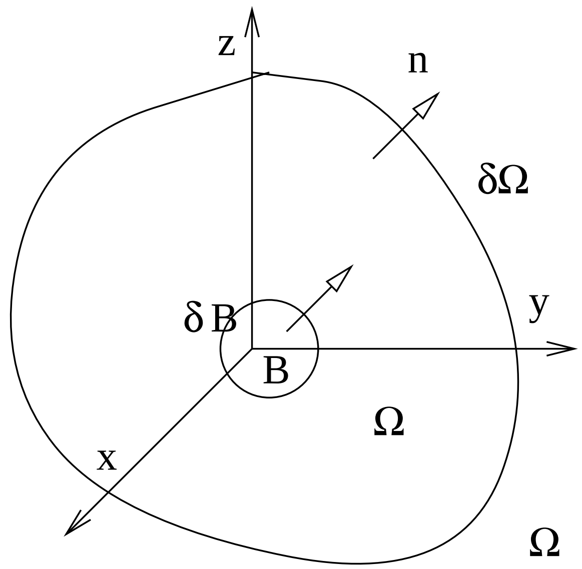

Figure 4.2: A ball cut out at the origin.

$$\int_{\partial B} \left(\frac{u}{\epsilon^2} + \frac{1}{\epsilon}\frac{\partial u}{\partial r} \right) \epsilon^2\, d\omega = \int_{\partial\Omega} \left[u\frac{\partial(1/r)}{\partial n} - \frac{\partial u}{\partial n}(1/r) \right] d\sigma,$$

since $r = \epsilon$ on $\mathcal{B}$ and $d\sigma = \epsilon^2\, d\omega$. The integrals converge uniformly, so

$$\lim_{\epsilon\to 0} \int_{\partial B} \left[u(x+\epsilon\omega) + \epsilon\frac{\partial u}{\partial r}(x+\epsilon\omega) \right] d\omega = u(x)\int_{\partial B} d\omega = 4\pi u(x).$$

This gives the formula

$$u(x) = -\frac{1}{4\pi}\int_{\partial\Omega}\left(u\frac{\partial(1/r)}{\partial n} - \frac{1}{r}\frac{\partial u}{\partial n}\right) d\sigma \tag{4.5}$$

Indeed, the above argument shows that

$$\int_B u\Delta(1/r)\,dx = -\int_{\partial B}\left(u\frac{\partial(1/r)}{\partial r} - \frac{\partial u}{\partial r}(1/r)\right) d\sigma = 4\pi u(x),$$

or

$$\Delta\left(\frac{1}{4\pi r}\right) = \delta(r), \quad r = |x|,$$

is just the Dirac delta function from Section 2.3. Any function satisfying this equation is called a fundamental solution of the potential equation in three dimensions.

We know from the Dirichlet and Neumann problems that it is enough to specify either the solution or its normal derivative on the boundary. This means that 4.5 is not exactly the formula that we are looking for since it requires both. However, we can add to $1/4\pi r$ any harmonic function. If we add harmonic function w such that either $G = \frac{1}{4\pi r} + w = 0$ or $\frac{\partial G}{\partial n} = 0$ on the boundary, then we obtain the correct formula.

Definition 4.5.1 *$G(x, y)$ is the Green's function for a region Ω if it has the following properties:*

1. *$G - 1/4\pi|x-y|$ is a harmonic function on Ω.*
2. *Dirichlet BC, $G = 0$ on $\partial\Omega$.*
3. *Neumann BC, $\frac{\partial G}{\partial n} = 0$ on $\partial\Omega$.*

Now we apply the argument above and obtain the formula giving the solution of the Dirichlet BVP on Ω,

$$u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial n}(x, y)\,d\sigma(y) \tag{4.6}$$

and the formula for the solution of the Neumann BVP on Ω,

$$u(x) = \int_{\partial\Omega}\frac{\partial f}{\partial n}(y)G(x, y)\,d\sigma(y). \tag{4.7}$$

Chapter 5

Green's Functions

5.1 Kelvin's Method of Images

We will now show how to construct Green's functions by way of examples. The first one is,

Example 5.1.1 *The Green's function for the half-space.*

The Dirichlet BVP for the upper-half of space is,

$$\Delta u = 0, \quad x, y \in \mathbb{R},\ z \in \mathbb{R}^+.$$

BC,

$$u(x, y, 0) = g(x, y),\ x, y \in \mathbb{R},$$
$$|u| \to 0 \text{ if } |(x, y, z)| \to \infty.$$

We solve this BVP by finding a Green's function, see Definition 4.5.1, $G = \frac{1}{4\pi|y-x|} + w$, which is a solution of the BVP

$$\Delta \left(G(x, y) - \tfrac{1}{4\pi|y-x|} \right) = 0$$
$$G(x, y) = 0, \quad y = (x', y', 0)$$
$$|G| \to 0 \text{ as } |x| \text{ or } |y| \to \infty.$$

The simplest way to solve this problem is by employing physical reasoning. The electric potential at a point y due to an electric charge

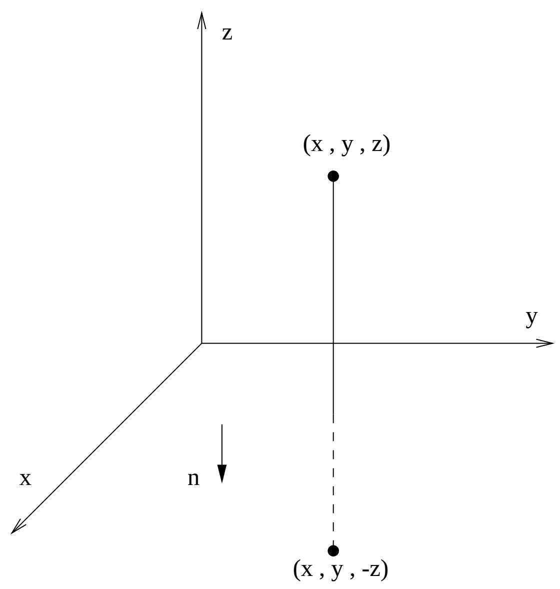

Figure 5.1: Reflection in the horizontal plane.

located at x, is $1/4\pi|y-x|$, if $q/\epsilon_0 = 1$, see Section 4.3. How can we make the potential vanish in the plane $z = 0$? The answer is by placing an equally large negative charge at the point $\bar{x} = (x, y, -z)$ which is the reflection of the point x (a vector) in the plane $z = 0$, see Figure 5.1. $w = \frac{1}{4\pi|y-\bar{x}|}$ is a harmonic function in the upper half-space because its singularity is located in the lower half-space.

Solution: The Green's function is

$$G(x,y) = \frac{1}{4\pi}\frac{1}{|y-x|} - \frac{1}{4\pi}\frac{1}{|y-\bar{x}|},$$

$y = (x', y', z')$ and on the $x' - y'$ plane $y = (x', y', 0)$,

$$\begin{aligned} G &= \frac{1}{4\pi}\left[\frac{1}{((x-x')^2+(y-y')^2+z^2)^{1/2}} - \frac{1}{((x-x')^2+(y-y')^2+z^2)^{1/2}}\right] \\ &= 0. \end{aligned}$$

The normal derivative is, see Figure 5.1,

$$\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial z'}$$

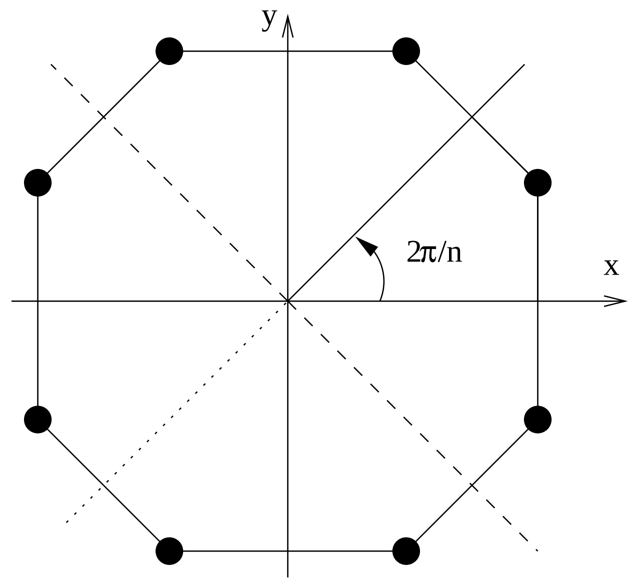

Figure 5.2: Reflection in a wedge.

$$
\begin{aligned}
&= \frac{1}{4\pi}\Bigg[\frac{z'-z}{((x-x')^2+(y-y')^2+(z'-z)^2)^{3/2}} \\
&\qquad - \frac{z'+z}{((x-x')^2+(y-y')^2+(z'-z)^2)^{3/2}}\Bigg] \\
&= \frac{-z}{2\pi((x-x')^2+(y-y')^2+z^2)^{1/2}}
\end{aligned}
$$

at $z'=0$. Thus the solution of the Dirichlet problem is

$$
u(x,y,z) = \frac{z}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{g(x',y')\,dx'dy'}{((x-x')^2+(y-y')^2+z^2)^{3/2}}. \tag{5.1}
$$

It is called the *Poisson formula* for the (upper) half-space.

The method we used is called Kelvin's method of images. We can use similar reasoning to find the Green's funciton for a wedge-shaped region if the angle of the wedge is $2\pi/n$, $n \in \mathbb{Z}^+$, see Figure 5.2, or a sphere. The reflection of a point (x,y,z), $x^2+y^2+z^2=r^2$, in the surface of a sphere with radius R is $\left(\frac{R^2x}{r^2}, \frac{R^2y}{r^2}, \frac{R^2z}{r^2}\right)$, see Figure 5.3.

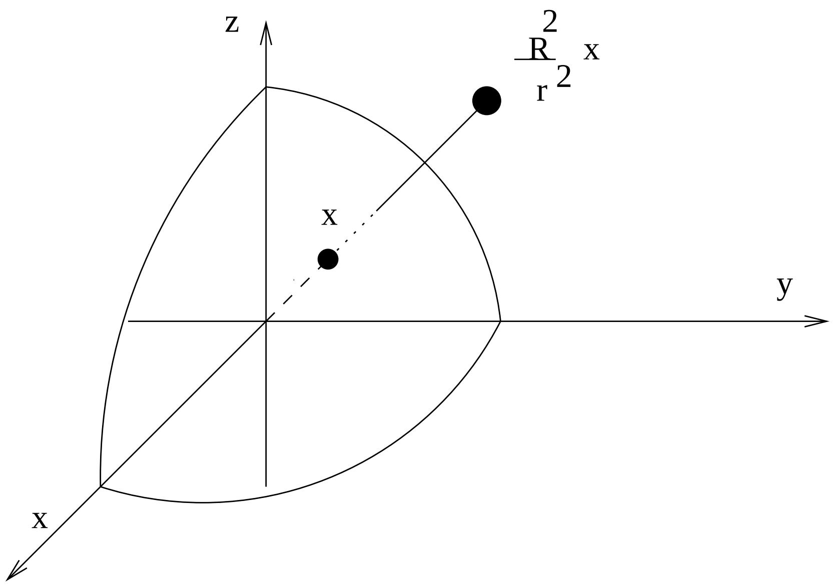

Figure 5.3: Reflection in a sphere.

Example 5.1.2 *Poisson's Formula for the Disk.*

We use Kelvin's method of images to place a source at (x, y) and another one at its reflection in a circle of radius R: $\frac{R^2}{r^2}(x, y)$, where $x^2 + y^2 = r^2$, see Figure 5.4. It makes no difference for the Green's functions, $\log |(x, y)| = \log |z|$, $z = x + iy$, whether we place the point in the real or the complex plane. Now the Green's function for the Dirichlet problem is the difference between the two sources

$$G((x, y), (x', y')) = \frac{1}{2\pi}\mathrm{Re}[\log(z' - z) - \log(z' - z^*)],$$

where $z' = \mathrm{Re}^{i\phi}$, $z = re^{i\theta}$ and $z^* = \frac{R^2}{r^2}z$. The normal derivative of the Green's function is

$$\frac{\partial G}{\partial n} = \frac{\partial G}{\partial R} = \frac{1}{2\pi}\mathrm{Re}\left(\frac{z'}{z' - z} - \frac{z'}{z' - z^*}\right)\frac{1}{R}.$$

Now,

$$\frac{z'}{z' - z} - \frac{z'}{z' - z^*} = \frac{(z - z^*)z'}{(z' - z)(z' - z^*)} = -\frac{(z\bar{z} - z'\bar{z}')}{(z' - z)(\bar{z}' - \bar{z})},$$

because $z^* = \frac{z'\bar{z}'}{\bar{z}}$,

$$= \frac{R^2 - r^2}{(R - r\,e^{i(\theta-\phi)})(R - re^{-i(\theta-\phi)})}$$

$$= \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta-\phi) + r^2}\,.$$

Thus

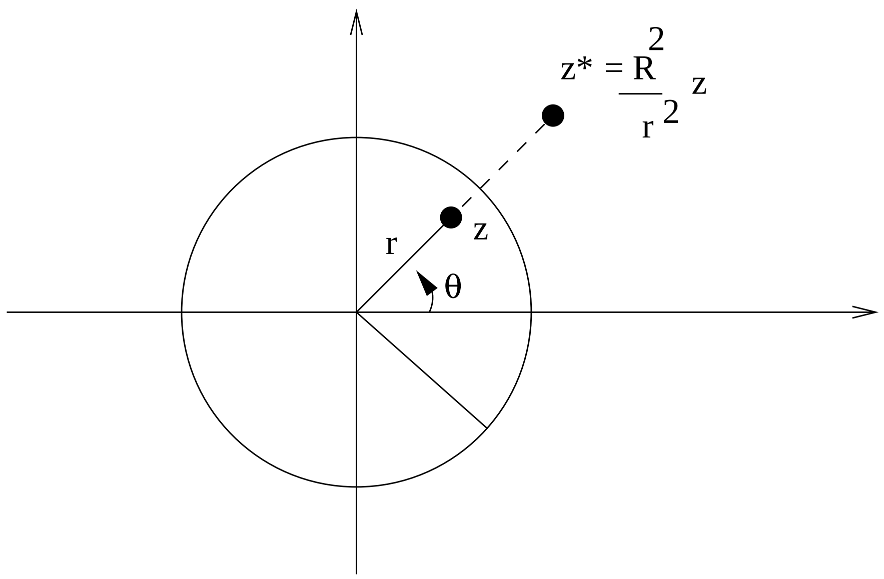

Figure 5.4: Reflection in a circle.

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\frac{\partial G}{\partial R} R\,d\phi \\ &= \frac{(R^2 - r^2)}{2\pi}\int_0^{2\pi} \frac{f(\phi)\,d\phi}{R^2 - 2Rr\cos(\theta-\phi) + r^2}. \end{aligned} \tag{5.2}$$

The last formula that we have expressed in polar coordinates is called *Poisson's formula* for the disk of radius R.

Exercises 5.1.3

5.1 Use the Green's functions

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}| - \log|\mathbf{x}-\bar{\mathbf{y}}|$$

$\mathbf{x} = (x, y),\ \mathbf{y} = (x', y'),\ \bar{\mathbf{y}} = (x', -y')$ to solve the Dirichlet BVP:

$$\Delta u = 0, \quad x \in \mathbb{R},\ y \in \mathbb{R}^+,$$

BC,

$$u(x, 0) = e^{-x^2}, \quad x \in \mathbb{R}.$$

5.2 Show that the Neumann BVP,

$$\Delta u = 0, \quad x \in \mathbb{R},\ y \in \mathbb{R}^+,$$

BC,

$$\frac{\partial u}{\partial x}(x, 0) = xe^{-x^2}, \quad x \in \mathbb{R},$$

has a solution and then solve it.

5.3 Find the Green's function for the mixed BVP,

$$\Delta u = 0, \quad x \in \mathbb{R}^+,\ y \in \mathbb{R}^+,$$

BC,

$$\frac{\partial u}{\partial y}(x, 0) = -g(x), \quad u(0, y) = f(y), \quad x, y \in \mathbb{R}^+.$$

5.2 Poisson's Equation

The inhomogeneous potential equation with a source term

$$\Delta u = F(x),$$

is called Poisson's equation. It turns out that if we can solve the equation

$$\Delta u = \delta(x-y),$$

then we can also solve Poisson's equation. In Section 4.5 we saw that

$$\Delta\left(\frac{1}{4\pi|x|}\right) = \delta(x).$$

The solution is simply the convolution of $\frac{1}{4\pi|x|}$ and $F(x)$, namely

$$u(x) = \frac{1}{4\pi|x|} * F(x) = \frac{1}{4\pi}\int_\Omega \frac{F(y)}{|x-y|}\,dy,$$

where the last integral defines the convolution operator $*$. Then

$$\begin{aligned}\Delta u &= \int_\Omega F(y)\Delta\left(\frac{1}{4\pi|x-y|}\right)dy\\ &= \int_\Omega F(y)\delta(x-y)\,dy = F(x).\end{aligned}$$

The problem is that the integral

$$\int_\Omega F(y)\Delta\left(\frac{1}{4\pi|x-y|}\right)dy$$

is not convergent. Thus we cannot operate with the Laplacian inside the integral and that step is not justified. We now fix this by using the methods we used in Section 4.5 to get the solutions of the Dirichlet and the Neumann problems.

Theorem 5.2.1 *If $F(x) \in C^2(\Omega)$ has two derivatives on a bounded region Ω then*

$$u(x) = \frac{1}{4\pi}\int_\Omega \frac{F(y)}{|x-y|}\,dy$$

is a solution of Poisson's equation in Ω.

Proof. We use the same method as in the previous section and cut a ball $\mathcal{B}$, with center x and radius ϵ, out of the region Ω. Then if Ω' is the complement of $\mathcal{B}$ in Ω, we have $\Omega = \Omega' \cup \mathcal{B}$, see Figure 4.2. This allows us to express u as

$$u(x) = \lim_{\epsilon \to 0} \frac{1}{4\pi} \int_{\Omega'} F(y) \frac{1}{|x-y|}\, dy$$

We cannot just operate with Δ under the integral sign on the left of this formula because the resulting integral does not converge. If we could the result would be the delta function and it would pick out the value $F(x)$ that we are after. However, we obtain the same result by two integrations by parts or rather two applications of the divergence theorem. First we differentiate u with respect to the first coordinate x_1 of $x = (x_1, x_2, x_3)$,

$$\partial_{x_1} u(x) = \lim_{\epsilon \to 0} \frac{1}{4\pi} \int_{\Omega'} F(y) \frac{\partial}{\partial x_1} \frac{1}{|x-y|}\, dy.$$

Then the divergence theorem is used to integrate the last integral by parts, namely if $h = (F, 0, 0)$ is a three-vector, then

$$\begin{aligned}
\int_{\Omega'} F(y) \frac{\partial}{\partial x_1} \frac{1}{|x-y|}\, dy \quad &= \quad -\int_{\Omega'} h \cdot \nabla_y \frac{1}{|x-y|}\, dy = \\
-\int_{\partial\Omega'} F(y) n_1 \frac{1}{|x-y|}\, d\sigma \quad &+ \quad \int_{\Omega'} \frac{\partial}{\partial y_1} F(y) \frac{1}{|x-y|}\, dy,
\end{aligned}$$

where $y = (y_1, y_2, y_3)$, $n = (n_1, n_2, n_3)$, and we have used that

$$\frac{\partial}{\partial x_1} \frac{1}{|x-y|} = -\frac{\partial}{\partial y_1} \frac{1}{|x-y|}.$$

The upshot is that

$$\begin{aligned}
\partial_{x_1} u(x) = -\int_{\partial\Omega} F(y) n_1 \frac{1}{|x-y|}\, d\sigma \quad &+ \quad \int_{\Omega} \frac{\partial}{\partial y_1} F(y) \frac{1}{|x-y|}\, dy \\
\lim_{\epsilon \to 0} [-\int_{\partial B} F(y) n_1 \frac{1}{|x-y|}\, d\sigma \quad &+ \quad \int_{B} \frac{\partial}{\partial y_1} F(y) \frac{1}{|x-y|}\, dy].
\end{aligned}$$

The two limits are easily computed

$$\lim_{\epsilon\to 0}\int_{\partial B} F(y)n_1\frac{1}{|x-y|}\,d\sigma = \lim_{\epsilon\to 0}\int_{\omega} F(x-\epsilon\omega)n_1\epsilon\ d\omega = 0,$$

and

$$\lim_{\epsilon\to 0}\int_{B}\frac{\partial}{\partial y_1}F(y)\frac{1}{|x-y|}\,dy = \lim_{\epsilon\to 0}\int_0^{\epsilon}\int_{\omega}\frac{\partial}{\partial y_1}F(x-s\omega)sds\ d\omega = 0,$$

where $\int_\omega d\omega$ denotes the integral over the two- or three-dimensional angle variables. Thus

$$\partial_{x_1}u(x) = -\int_{\partial\Omega}F(y)n_1\frac{1}{|x-y|}\,d\sigma + \int_{\Omega}\frac{\partial}{\partial y_1}F(y)\frac{1}{|x-y|}\,dy,$$

where both integrals are nicely convergent. Now we differentiate again with respect to x_1,

$$\partial^2_{x_1}u(x) = -\int_{\partial\Omega}F(y)n_1\frac{\partial}{\partial x_1}\frac{1}{|x-y|}\,d\sigma + \int_{\Omega}\frac{\partial}{\partial y_1}F(y)\frac{\partial}{\partial x_1}\frac{1}{|x-y|}\,dy.$$

We express Ω again as a lim it of Ω' as $\epsilon\to 0$, and get

$$\begin{aligned}
\partial^2_{x_1}u(x) = -\int_{\partial\Omega}F(y)n_1\frac{\partial}{\partial x_1}\frac{1}{|x-y|}\,d\sigma &+ \lim_{\epsilon\to 0}\int_{\Omega'}\frac{\partial}{\partial y_1}F(y)\frac{\partial}{\partial x_1}\frac{1}{|x-y|}\,dy \\
= &- \int_{\partial\Omega}F(y)n_1\frac{\partial}{\partial x_1}\frac{1}{|x-y|}\,d\sigma \\
+\lim_{\epsilon\to 0}[\int_{\partial\Omega}F(y)n_1\frac{\partial}{\partial x_1}\frac{1}{|x-y|}\,d\sigma &+ \int_{\partial B}F(y)n_1\frac{\partial}{\partial x_1}\frac{1}{|x-y|}\,d\sigma \\
&- \int_{\Omega'}F(y)\frac{\partial^2}{\partial x_1\partial y_1}\frac{1}{|x-y|}\,dy],
\end{aligned}$$

by another application of the divergence theorem, with $h = (\frac{\partial F}{\partial y_1}, 0, 0)$. The first two integral cancel and we get

$$\partial^2_{x_1}u(x) = \lim_{\epsilon\to 0}[\int_{\partial B}F(y)n_1\frac{\partial}{\partial x_1}\frac{1}{|x-y|}\,d\sigma + \int_{\Omega'}F(y)\frac{\partial^2}{\partial x_1^2}\frac{1}{|x-y|}\,dy],$$

where again we have used that

$$\frac{\partial}{\partial x_1}\frac{1}{|x-y|} = -\frac{\partial}{\partial y_1}\frac{1}{|x-y|}.$$

Differentiation with respect to the other two coordinates x_2 and x_3 gives similar results and adding the analogous expressions for $\partial^2_{x_2} u(x)$ and $\partial^2_{x_3} u(x)$ to the expression above produces

$$\Delta u = lim_{\epsilon\to 0}[\int_{\partial B} F(y) n \cdot \nabla \frac{1}{|x-y|}\, d\sigma + \int_{\Omega'} F(y)\Delta \frac{1}{|x-y|}\, dy].$$

However, $\Delta \frac{1}{|x-y|} = 0$ in Ω', by Example 4.1.1 and $n\cdot\nabla = \frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$ on ∂B, so finally we get that

$$\begin{aligned}\Delta u &= -lim_{\epsilon\to 0}\int_{\partial B} F(y)\frac{\partial}{\partial r}(\frac{1}{r}) d\sigma \\ &= lim_{\epsilon\to 0}\int_{\omega} F(x-\epsilon\omega) d\omega \\ &= 4\pi F(x). \qquad \square\end{aligned}$$

We can let the region Ω expand to infinity if we add the hypothesis that F/r and $\nabla F/r$ are integrable. This gives the solution to Poisson's equation in $\mathbb{R}^3$,

$$u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{F(y)}{|x-y|}\, dy \tag{5.3}$$

The solution in Theorem 5.2.1 satisfies neither a Dirichlet nor a Neumann BC, to get a solution of Poisson's equation that satisfies these conditions, we must convolve with a Green's function from Section 4.5,

$$u(x) = \int_\Omega G(x,y)F(y)\, dy.$$

Finally the superposition principle 2.1.2 gives the solution of Poisson's equation with Dirichlet BC,

$$\begin{aligned}u(x) &= -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial n}(x,y)\, d\sigma(y) \\ &+ \int_\Omega G(x,y)F(y)\, dy, \end{aligned} \tag{5.4}$$

by Equation 4.6.

The solution of Poisson's equation with Neumann BC is,

$$\begin{aligned}u(x) &= \int_{\partial\Omega} \frac{\partial f}{\partial n}(y) G(x,y)\, d\sigma(y) \\ &+ \int_\Omega G(x,y)F(y)\, dy, \end{aligned} \tag{5.5}$$

by the superposition principle 2.1.2 and Equation 4.7.

Exercises 5.2.2

5.4 Solve the inhomogeneous Neumann BVP:

$$\begin{aligned}\Delta u &= \frac{1}{1+(x^2+y^2)^{1/2}}, \quad x \in \mathbb{R},\ y \in \mathbb{R}^+,\\ \frac{\partial u}{\partial y} &= xe^{-|x|}, \quad x \in \mathbb{R}.\end{aligned}$$

How many derivatives does the solution have for $y > 0$?

5.5 Show that the Neumann BVP,

$$\Delta u = 0, \quad r < 3, \quad 0 \leq \theta < 2\pi,$$

BC,

$$\frac{\partial u}{\partial r}(3, \theta) = 2, \quad 0 \leq \theta < 2\pi,$$

has no solution.

5.3 Green's Functions and Conformal Maps

The method of images can be used to construct Green's functions for regions whose boundaries are planes or lines in two dimensions. It can also be used to construct Green's functions when the boundary is a sphere or a circle as illustrated in Section 5.1. It is more difficult to prove the existence of the Green's function for a general non-linear boundary. If the boundary is extremely complicated then the Dirichlet problem may not even be solvable, see Courant and Hilbert, part II [4], for such an example. The reason is that the solution may not be able to assume its boundary data in any reasonable way.

Methods from *Functional Analysis, the Dirichlet Principle*, see F. John [6], or the theory of subharmonic functions, see Conway [3], can be used to prove the existence of the solutions to the Dirichlet and

Neumann problems in three and higher dimensions. In two dimension we can use the methods of complex analysis. In particular we can use the following theorem to prove the existence of the Green's function for every simply connected region.

Theorem 5.3.1 (The Riemann Mapping Theorem) *Every simply connected region in the complex plane can be mapped conformally onto the unit disk, such that the boundary of the region is mapped onto the boundary of the unit disk.*

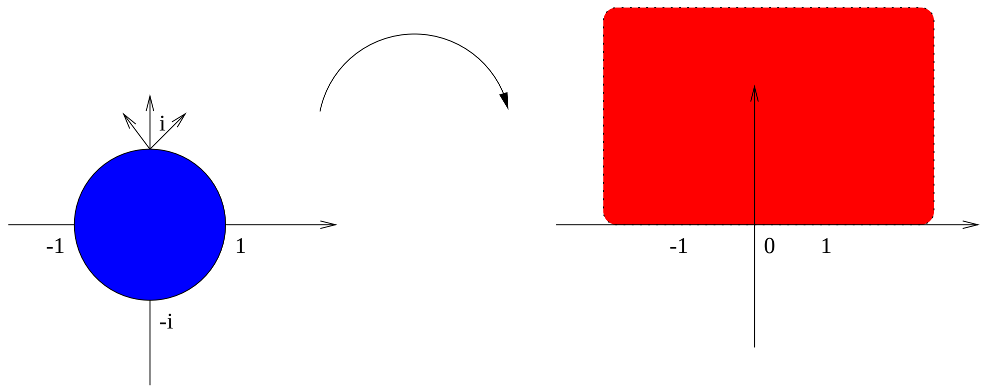

Figure 5.5: The map of the disk to the upper half-plane.

The proof of the Riemann Mapping Theorem can be found in L. Ahlfors [1]. A conformal map is an analytic function that preserves the angles between two vectors in the complex plane, while stretching or contracting the modulus of each vector.

Example 5.3.2

We have the conformal map that maps the unit disk onto the upper half-plane. This map is of the form

$$f(z) = \frac{az+b}{cz+d}, \quad \text{with } ad - cb \neq 0,$$

and is called a fractional linear or a Möbius transformation. It can be constructed by composing three maps, namely, translation, dilation and inversion,

$$z \to z+b, \quad z \to az, \quad z \to 1/z.$$

The last map will send the origin to infinity and we have to start by sending a boundary point to infinity, in order to map a bounded region, the unit disk, onto an unbounded region, the upper half-plane. We choose the point $z = i$ for this purpose and send it to infinity by the map $z \to 1/(z-i)$. Next we send $-i$ to the origin by myltiplying by $z+i$ and get the map

$$z \to \frac{z+i}{z-i}.$$

Now for fractional linear maps we need to check that the boundary maps to the boundary. It turnes out that circles and lines are mapped to either circles or lines. In addition we must check that one point on the interior of the unit disk maps onto the interior of the upper half-plane. Then the rest of the interior follows by continuity. We check $z = 0$, $\frac{0+i}{0-i} = -1$, which still lies on the boundary, so we rotate clockwise by $90°$ by multiplying by $e^{-i\pi/2} = -i$. This gives the map

$$\begin{gathered} h(z) = -i\left(\tfrac{z+i}{z-i}\right), \\ h(+i) = \infty, \quad h(-i) = 0, \\ h(1) = -i\tfrac{(1+i)}{(1-i)} = -i\tfrac{x+2i-x}{2} = 1 \\ h(-1) = -i\tfrac{-1+i}{-(1+i)} = -i(1-2i-1) = -1, \end{gathered}$$

and

$$h(0) = -i\frac{i}{-i} = i,$$

so the boundary is mapped to the boundary and the interior onto the interior. We have torn open the unit disk at i, sent the two loose ends to infinity, jerked $-i$ up to the origin and spilled the inside of the unit disk onto the upper half plane, see Figure 5.5.

The Riemann Mapping Theorem implies that we can solve the Dirichlet and Neumann problems for every simply connected region in the plane because it gives us a method for constructing the Green's function for the region. We just have to recall from Section 3.5 that the Dirichlet Green's function for the upper half-plane was,

$$G((x,y),(x',y'))$$

$$
\begin{aligned}
&= \frac{1}{2\pi}[\log|(x,y)-(x',y')| - \log|(x,y)-(x',-y')|] \\
&= \frac{1}{2\pi}[\log|z-z'| - \log|z-\bar{z}'|],
\end{aligned}
$$

where $z = x + iy,\ z' = x' + iy'$ and $\bar{z}'$ denotes complex conjugation. Then let h be a map from a simply connected region Ω onto the upper half-plane. We can always find such a map by first mapping the region to the unit disk by the Riemann Mapping Theorem 5.3.1 and then mapping the unit disk onto the half-plane by the map in Example 5.3.2. h is the composition of the two maps and the Green's function becomes

$$
G((x,y),(x',y')) = \frac{1}{2\pi}[\log|h(z)-h(z')| \pm \log|h(z)-\overline{h(z')}|], \quad (5.6)
$$

where $+$ gives the solution of the Dirichlet and the $-$ the solution of the Neumann problems.

Example 5.3.3

We find the Green's function that solves the Dirichlet problem for the quarter plane $x \geq 0,\ y \geq 0$, see Figure 5.6, by use of a conformal mapping. First the conformal map that maps the quarter-plane in the

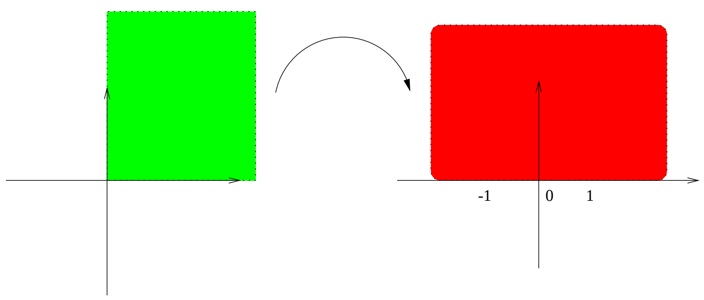

Figure 5.6: The map of the quarter-plane to the upper half-plane.

first quadrant onto the upper half-plane is $h(z) = z^2$. The imaginary axis $z = ir$ gets mapped onto the negative real axis $w = -r^2,\ r > 0$, but the positive real axis is mapped onto itself. By the above formula the Green's function for the Dirichlet problem is,

$$\begin{aligned} G((x,y),(x',y')) &= \frac{1}{2\pi}[\log|z^2 - z'^2| - \log|z^2 - \bar{z}'^2|] \\ &= \frac{1}{2\pi}\log\left[\frac{(x^2-y^2-x'^2+y'^2)^2+4(xy-x'y')^2}{(x^2-y^2-x'^2+y'^2)^2+4(xy+x'y')^2}\right]^{1/2}, \end{aligned}$$

where $z = x + iy$ and $z' = x' + iy'$.

Exercises 5.3.4

5.6 Find a conformal map of the semi-disk, $x^2 + y^2 \leq 1,\ y \geq 0$, onto the upper half-plane and use it to solve the Dirichlet BVP:

$$\Delta u = 0, \quad x^2 + y^2 \leq 1,\ y > 0,$$

BC,

$$\begin{aligned} u(x,0) &= f(x), & -1 \leq x \leq 1, \\ u(1,\theta) &= g(\theta), & 0 \leq \theta < \pi. \end{aligned}$$

Hint: First use the map $1/(z-1)$ of the semi-disk.

5.7 Find the (Dirichlet) Green's function for the region $0 \leq x \leq 1,\ y \geq 0$, by use of a conformal map.

Hint: This map is a trigonometric function.

5.4 Additional Exercises for the Potential Equation

5.8 Prove or disprove that the product of two harmonic functions is harmonic.

5.9 A rocket is a cylinder with radius 2 meters. Find the electric potential inside the rocket if it is $u(2,\theta) = \sin(\theta)$ on all circles that are the circumference of a cross-section of the cylinder.

5.10 Show that if the electric field is $\frac{\partial u}{\partial r}(2, \theta) = \cos(\theta)$ on the surface of the rocket, instead of the electric potential in Problem 3, then the Neumann BVP has a solution.

5.11 Compare the properties of the solutions of the wave, the heat and the potential equation. Discuss what kind of problems to pose, characteristics, whether initial and boundary data becomes smoother, propagation speed, etc. Discuss which initial or boundary value problems are well-posed, i.e., existence, uniqueness and stability.

Chapter 6

Higher-Dimensional Fourier Transform and Wave, Heat and Schrödinger Equations

6.1 The Fourier Transform

The inner product of two vectors $\mathbf{k} = (k_1, \ldots, k_n)$ and $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ is defined to be

$$\mathbf{k} \cdot \mathbf{x} = k_1 k_2 + k_2 x_2 + \ldots + k_n x_n.$$

The Fourier transform $\mathcal{F}(f)$ of a function $f(\mathbf{x})$ of n variables can then be written

$$\hat{f}(\mathbf{k}) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} \, d\mathbf{x},$$

where $\int_{\mathbb{R}^n} d\mathbf{x}$ denotes the iterated integral $\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} dx_1 dx_2 \ldots dx_n$. This assumes that f is integrable on $\mathbb{R}^n$, but actually the Fourier transform can be defined for functions which are only square integrable, that is f^2 is integrable, or less. The inverse Fourier transform is defined to be

$$\check{g}(\mathbf{x}) = \int_{\mathbb{R}^n} g(\mathbf{k}) e^{2\pi \mathbf{i} \mathbf{k} \cdot \mathbf{x}} \, d\mathbf{k},$$

assuming g is integrable. These two operations are inverses

$$f(\mathbf{x}) = (\hat{f})^{\vee}$$

if we can make sense of both $\hat{f}(\mathbf{k})$ and $(\hat{f})^{\vee}(\mathbf{x})$.

Convolution, introduce in Section 5.2, can also be defined in higher dimensions

$$f * g(\mathbf{x}) = \int_{\mathrm{I\!R}^n} f(\mathbf{y}-\mathbf{x})g(\mathbf{y})\,d\mathbf{y},$$

if one of f and g is integrable and the other one is bounded. We will need the following properties of the Fourier transform and convolution, the most important being that the Fourier transform of a convolution is the product of the Fourier transforms of the two functions forming the convolution.

Property 6.1.1 The Fourier Transform
Let f be an integrable function on $\mathrm{I\!R}^n$, then

a. $\mathcal{F}(f(\mathbf{x}-\mathbf{y})) = e^{-2\pi i\mathbf{k}\cdot\mathbf{y}}\hat{f}(\mathbf{k})$

b. $\mathcal{F}\left(\frac{\partial f}{\partial x_j}\right) = 2\pi i k_j \hat{f}(\mathbf{k})$

c. $\mathcal{F}(x_j f(\mathbf{x})) = 2\pi i \frac{\partial \hat{f}}{\partial k_j}$

d. $\widehat{f * g} = \hat{f}\hat{g}$

e. $\mathcal{F}(f(|\mathbf{x}|)) = \hat{f}(|\mathbf{k}|)$.

6.2 The Wave Equation in Dimensions Two and Three

We will now compute the solution of the wave equation in dimensions two and greater using the Fourier transform. Consider the initial value problem

$$u_{tt} = c^2\Delta u, \qquad x \in \mathrm{I\!R}^n,\ t \in \mathrm{I\!R},$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

We take the Fourier transform of these three equations and get

$$\begin{aligned} \hat{u}_{tt} + 4\pi^2c^2|k|^2\hat{u} = 0, &\quad k \in \mathrm{I\!R}^n,\ t \in \mathrm{I\!R} \\ \hat{u}(k,0) = \hat{f}(k), &\qquad \hat{u}_t(k,0) = \hat{g}(k). \end{aligned}$$

The term

$$\widehat{\Delta u} = \int_{\mathrm{I\!R}^n} \Delta u e^{-2\pi i k\cdot x}\,dx = -4\pi^2|k|^2\hat{u}$$

is evaluated by integration by parts just as the one dimensional case in Section 2.2. The ODE IVP is just that of the harmonic oscillator and it is easily solved,

$$\hat{u}(k,t) = \hat{f}(k)\cos(2\pi c|k|t) + \hat{g}(k)\frac{\sin(2\pi c|k|t)}{2\pi c|k|}.$$

We need to apply the inverse Fourier transform

$$u(x,t) = \int_{-\infty}^{\infty} \hat{u}(k,t)e^{2\pi i k\cdot x}\,dk$$

to obtain the solution. In one dimension, because both $\cos(y)$ and $\sin(y)/y$ are even functions, we can omit the absolute value (length) sign on k and get

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty}\left[\hat{f}(k)\cos(2\pi ckt) + \hat{g}(k)\frac{\sin(2\pi ckt)}{2\pi ck}\right]e^{2\pi ikx}\,dk \\ &= \int_{-\infty}^{\infty}\hat{f}(k)\left[\frac{e^{2\pi i(x+ct)k}}{2} + \frac{e^{2\pi i(x-ct)}}{2}\right]dk \\ &\qquad + \int_{-\infty}^{\infty}\hat{g}(k)\left[\frac{e^{2\pi i(x+ct)}}{4\pi ick} - \frac{e^{2\pi i(x-ct)}}{4\pi ick}\right]dk \end{aligned}$$

by use of the identities

$$\cos(y) = \frac{e^{iy}+e^{-iy}}{2}, \quad \sin(y) = \frac{e^{iy}-e^{iy}}{2i},$$

so

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty}\hat{f}(k)\frac{e^{2\pi i(x+ct)k}}{2}dk + \int_{-\infty}^{\infty}\hat{f}(k)\frac{e^{2\pi i(x-ct)k}}{2}\,dk \\ &\qquad + \int_{-\infty}^{\infty}\hat{g}(k)\int_{x-ct}^{x+ct}\frac{e^{2\pi iyk}}{2c}\,dy\,dk \\ &= \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy, \end{aligned}$$

if we switch the order of the last two integration. This is permissible if $\int_{\infty}^{\infty} |\hat{g}(k)|\, dk < \infty$, and we have rederived the d'Alembert solution from Section 1.3.

The computation is more complicated in two and three dimensions. We start with three dimensions and notice first that

$$\cos(2\pi c|k|t) = \frac{d}{dt}\frac{\sin(2\pi c|k|t)}{2\pi c|k|}$$

so that if we can evaluate

$$\int_{-\infty}^{\infty} \hat{g}(k)\frac{\sin(2\pi c|k|t)}{2\pi c|k|}e^{2\pi ik\cdot x}dk = W(t) * g$$

then

$$\int_{-\infty}^{\infty} \hat{f}(k)\cos(2\pi c|k|)e^{2\pi k\cdot x}dk = \frac{\partial}{\partial t}W(t) * f.$$

This means that it suffices to compute the inverse Fourier transform,

$$\int_{-\infty}^{\infty} \frac{\sin(2\pi c|k|t)}{2\pi c|k|}e^{2\pi ik\cdot x}dk = W(t,x)$$

and then use that

$$\widehat{W * g} = \hat{W}\hat{g}$$

so

$$(\hat{W}\hat{g})^{\vee} = W * g,$$

see Property 6.1.1 . By Property 6.1.1 the Fourier transform of a spherically symmetric function is also spherically symmetric. This leads us to look for a spherically symmetric function whose Fourier transform is $\frac{\sin(2\pi c|k|t)}{2\pi c|k|}$. Let $h(x)$ be a spherically symmetric function, then

$$\begin{aligned}
\hat{h}(|k|) &= \int_{\mathbb{R}^3} h(|x|)e^{2\pi ik\cdot x}dx \\
&= \int_0^{\infty} h(r)\int_0^{2\pi}\int_0^{\pi} e^{2\pi i\zeta r\cos\phi}r^2\sin\phi\, d\phi d\theta dr.
\end{aligned}$$

We have used here that $h(r)$ is invariant with respect to rotation so that we can choose $k = (0,0,\zeta)$ where $\zeta = |k|$. Then

$$k\cdot x = (0,0,\zeta)\cdot(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi) = \zeta r\cos\phi.$$

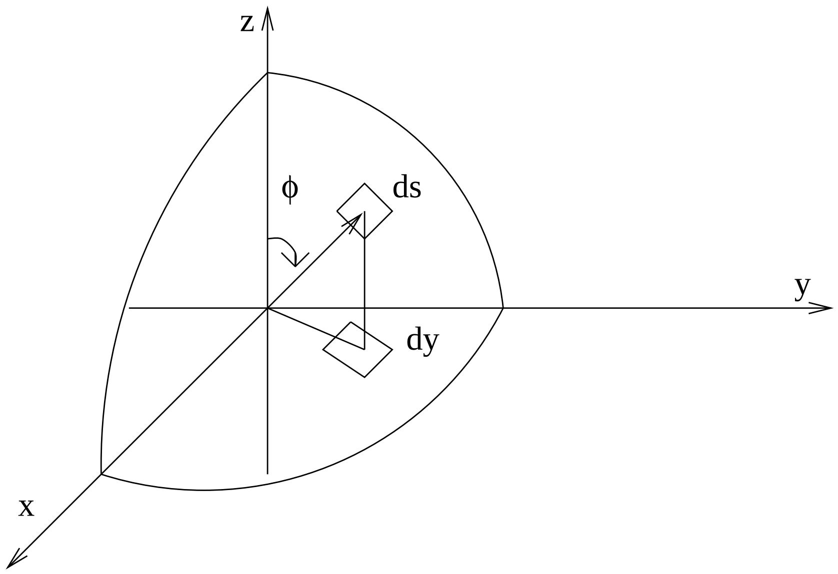

Figure 6.1: The projection of the sphere.

Now by integration

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi} e^{2\pi i\zeta r\cos\phi} r^2 \sin\phi d\phi d\theta &= 2\pi \frac{-e^{2\pi i\zeta r\cos\phi}}{2\pi i\zeta r}\Big|_0^{\pi} r^2 \\ &= \frac{2\sin(2\pi\zeta r)r}{\zeta}.\end{aligned}$$

Thus

$$\hat{h}(|k|) = 2\int_0^{\infty} h(r)\frac{\sin(2\pi\zeta r)r}{\zeta}dr.$$

The trick now is simply to let $h(r) = \delta(r-1)$ be the delta function, with a singularity at $r = 1$, strictly speaking we should take a sequence of smooth functions $h_n \to \delta$. Then

$$\hat{h}(\zeta) = \frac{2\sin(2\pi\zeta)}{\zeta},$$

see Section 2.3, and if $\zeta = c|k|t$, we get

$$\hat{h}(c|k|t) = 4\pi\left(\frac{\sin(2\pi c|k|t)}{2\pi c|k|}\right).$$

At this point we need to interpret this result to figure out what kind of distribution W, $\hat{W} = \frac{\hat{h}}{4\pi}$, is. It is clearly not a function. We have shown that if we divide by the factor $4\pi c^2 t$, then

$$\frac{1}{4\pi c^2 t}\int_{\mathbb{R}^3} \delta(|x| - ct) e^{-2\pi i k\cdot x}\, dx = \frac{1}{4\pi c^2 t}\int_{|x|=ct} e^{-2\pi i k\cdot x}\, d\sigma,$$

where $d\sigma = c^2 t^2\, d\omega = c^2 t^2 \sin\phi\, d\phi d\theta$, is the surface measure on a sphere with radius $|x| = ct$,

$$= \frac{t}{4\pi}\int_{|y|=1} e^{-2\pi i ctk\cdot y} d\sigma = t\frac{\sin(2\pi c|k|t)}{2\pi c|k|t} = \frac{\sin(2\pi c|k|t)}{2\pi c|k|}.$$

In other words $W(t)$ is just the surface measure on a sphere of radius ct,

$$W(t) = \frac{1}{4\pi c^2 t}\int_{|x|=ct} d\sigma$$

divided by $4\pi c^2 t$, or the integral of the radial delta function $\delta(r - ct)$. Now we can convolve W with a function

$$\begin{aligned} W * g &= \frac{1}{4\pi c^2 t}\int_{|x|=ct} g(x-y)\, d\sigma(y) \\ &= \frac{1}{4\pi c^2 t}\int_{|x-y|=ct} g(y)\, d\sigma \end{aligned}$$

by a change of variables. Then we obtain the solution of the wave equation in three dimensions,

$$u(x,t) = \frac{\partial}{\partial t}\frac{1}{4\pi c^2 t}\int_{|x-y|=ct} f(y)\, d\sigma + \frac{1}{4\pi c^2 t}\int_{|x-y|=ct} g(y)\, d\sigma. \tag{6.1}$$

This is the three dimensional analog of the d'Alembert formula, in Section 1.3.

The solution in two dimensions can be obtained by the three dimensional solution by Hadamard's method of descent. The observation is that if f and g are independent of x_3 then $u_{x_3x_3} = 0$ and the three-dimensional wave equation reduces to the two-dimensional one,

$$u_{tt} - c^2(u_{x_1x_1} + u_{x_2x_2}) = 0, \quad (x_1, x_2) \in \mathbb{R}^2,\ t \in \mathbb{R}.$$

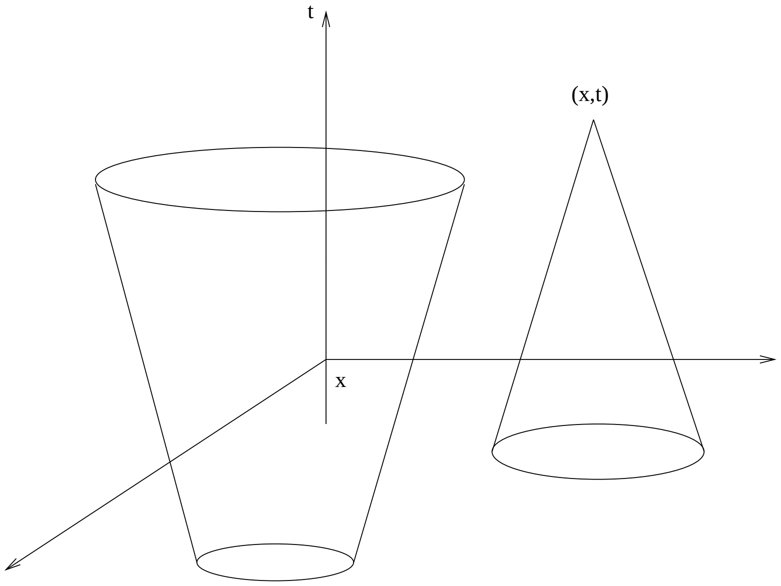

Figure 6.2: The domain of dependence of (x,t) and the range of influence of an x-sphere.

Moreover, the area element of the disk is

$$dy = \cos\phi \, d\sigma,$$

see Figure 6.1, but

$$ct\cos\phi = y_3 = (c^2t^2 - |y|^2)^{1/2}, \quad y = (y_1, y_2),$$

so the three dimensional solution reduces to

$$u(x,t) = \frac{1}{2\pi c}\frac{\partial}{\partial t}\int_{|x-y|\le ct}\frac{f(y)\,dy}{\sqrt{c^2t^2 - |y|^2}} + \frac{1}{2\pi c}\int_{|x-y|\le ct}\frac{g(y)\,dy}{\sqrt{c^2t^2 + |y|^2}}. \tag{6.2}$$

We get a factor of two on top because both the upper and lower hemisphere get mapped onto the disk.

Now we place an owl at x and make it hoot once loud and clear, the resulting solution of the wave equation, because sound just consists of airwaves, is supported on a spherical shell of radius ct. t is the time passed since the owl hooted. An observer positioned at y hears the hoot at time $|y-x|/c$, and then there will be silence again, see Figure 6.2. On the other hand in two dimensions, if we drop a stone in a pond at point x, then the resulting two-dimensional solution is supported in the disk $|y-x| \leq ct$. We see decaying ripples emanating from x forever after. In a two-dimensional world sound waves would persist and we are lucky that our world is three-dimensional.

6.3 The Higher-dimensional Heat and Schrödinger Equations

The derivation of the solutions to the heat and Schrödinger equations in dimensions two and greater is much simpler than that of the wave equation. Consider the IVP for the heat equations in $\mathbb{R}$,

$$\begin{aligned} u_t = \tfrac{1}{2}\Delta u, \qquad & x \in \mathbb{R}^n,\ t \in \mathbb{R}^+, \\ u(x,0) = f(x), \qquad & x \in \mathbb{R}^n. \end{aligned}$$

We take the Fourier transform of both sides of the heat equation to get an ODE

$$\hat{u}_t = -2\pi^2|k|^2\hat{u},$$

with the solution

$$\hat{u}(k,t) = \hat{u}(k,0)e^{-2\pi^2|k|^2 t},$$

where

$$\hat{u}(k,0) = \int_{-\infty}^{\infty} u(x,0)e^{-2\pi i k\cdot x}\,dx = \hat{f}(k),\ \ k \in \mathbb{R}^n,$$

is the Fourier transform of the initial temperature. The solution of the heat equation is obtained by the inverse Fourier transform

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} e^{-2\pi^2|k|^2 t}e^{2\pi i k\cdot x}\hat{f}(k)\,dk \\ &= \int_{-\infty}^{\infty} f(y)\int_{-\infty}^{\infty} e^{-2\pi^2|k|^2 t}e^{2\pi i k\cdot(x-y)}\,dk\,dy \end{aligned}$$

The inner integral is easily evaluated as a product of integrals because the exponential of a sum is the product of exponentials,

$$\int_{-\infty}^{\infty} e^{-2\pi^2|k|^2 t}e^{2\pi i k\cdot x}\,dk = \prod_{j=0}^{n}\int_{-\infty}^{\infty} e^{-2\pi^2 k_j^2 t}e^{2\pi i k_j x_j}\,dk.$$

The last integral was evaluated in Section 3.2 and it turned out to be the one-dimensional heat kernal. Thus the product is

$$\prod_{j=0}^{n} \frac{e^{-x_j^2/2t}}{\sqrt{2\pi t}} = \frac{e^{-|x|^2/2t}}{(2\pi t)^{n/2}}.$$

This give us the n-dimensional heat kernal

$$K(x,t) = \frac{e^{-|x|^2/2t}}{(2\pi t)^{n/2}},$$

and the solution of the heat equation in n dimensions,

$$u(x,t) = \int_{-\infty}^{\infty} f(y) \frac{e^{-|x-y|^2/2t}}{(2\pi t)^{n/2}}\, dy.$$

The IVP for the Schrödinger equation is

$$\begin{aligned} i\hbar \tfrac{\partial \psi}{\partial t} &= -\tfrac{\hbar^2}{2m}\Delta\psi, && x \in \mathbb{R}^n,\ t \in \mathbb{R}^+, \\ \psi(x,0) &= f(x), && x \in \mathbb{R}^n. \end{aligned}$$

We can solve it by making the change of variables,

$$t \to it,$$

in the solution of the heat equation, if we set $\hbar = 1 = m$,

$$\psi(x,t) = \int_{\mathbb{R}^n} \frac{f(y) e^{i|x-y|^2/2t}}{(2\pi i t)^{n/2}}\, dy.$$

Then the solution of the equation for arbitrary $\hbar$ and m is obtained by Remark 3.2.1,

$$\psi(x,t) \to \psi(x, \frac{\hbar}{m} t)$$

or

$$\psi(x,t) = \int_{\mathbb{R}^n} \frac{f(y) e^{i\hbar|x-y|^2/2mt}}{(2\pi\hbar i t/m)^{n/2}}\, dy.$$

is a solution to the original problem. The properties of ψ are more similar to the solutions of the wave equation than the solutions of the heat equation.

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