MATH 3B - Practice midterm solution key

1. Evaluate the following definite or indefinite integrals. You do not need to show work for these problems. (10 points)

(a) $\displaystyle\int \sqrt{x}+\frac{1}{\sqrt{x}}\,dx$

(b) $\displaystyle\int 3xe^{x^2}\,dx$

(c) $\displaystyle\int_0^{\pi/6} \tan x\,dx$

(d) $\displaystyle\int \frac{3}{2+2(x+3)^2}\,dx$

SOLUTION.

(a) $\displaystyle\int \sqrt{x}+\frac{1}{\sqrt{x}}\,dx = \frac{2}{3}x^{3/2}+2\sqrt{x}+C.$

(b) $\displaystyle\int 3xe^{x^2}\,dx = \frac{3}{2}e^{x^2}+C$

(c) $\displaystyle\int_0^{\pi/6} \tan x\,dx = \ln(2)-\frac{\ln(3)}{2}.$

(d) $\displaystyle\int \frac{3}{2+2(x+3)^2}\,dx = \frac{3}{2}\tan^{-1}(x+3)+C$

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2. Set up the limit of Riemann sums you would take to evaluate the following definite integrals. Do not evaluate the limit. (10 points)

(a) $\int_0^3 \sin(x)+2\,dx$

(b) $\int_{-1}^1 x^3+x^2-1\,dx$

(c) What important theorems allows us to take antiderivates instead of having to evaluate these terrible limits?

SOLUTION. First, recall the following theorem.

If f is integrable on $[a,b]$, then

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n}\sum_{i=1}^{n} f\left(a+i\tfrac{b-a}{n}\right).}$$

(a) Here we have $f(x)=\sin(x)+2$, $a=0$, and $b=3$. Since f is continuous we know it is integrable and so our theorem gives

$$\begin{aligned}\int_0^3 \sin(x)+2\,dx &= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\\ &= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\sin\left(\frac{3i}{n}\right)+2\right]\end{aligned}$$

(b) Here we have $f(x)=x^3+x^2-1$, $a=-1$, and $b=1$. Since f is continuous we know it is integrable and so our theorem gives

$$\begin{aligned}\int_{-1}^1 x^3+x^2-1\,dx &= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n} f\left(\frac{2i-n}{n}\right)\\ &= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\left(\frac{2i-n}{n}\right)^3+\left(\frac{2i-n}{n}\right)^2-1\right]\end{aligned}$$

(c) The fundamental theorem of calculus along with the fact that every continuous function admits an anti-derivative.

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3. Rewrite the following limit as a definite integral. (3 points)

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{5}{n}\ln\left(\frac{5i}{n}\right).$$

SOLUTION.

$$\int_0^5 \ln(x)\,dx = \lim_{n\to\infty}\frac{5-0}{n}\sum_{i=1}^{n}\ln\left(0+i\,\frac{5-0}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{5}{n}\ln\left(\frac{5i}{n}\right)$$

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4. Evaluate the following. (6 points)

(a) $\dfrac{d}{dx}\displaystyle\int_1^x e^{1/t} - 2t\,dt$

(b) $\dfrac{d}{dx}\displaystyle\int_{x^3}^2 e^{t^2} - \sin(t)\,dt$

SOLUTION.

(a) $\dfrac{d}{dx}\displaystyle\int_1^x e^{1/t} - 2t\,dt = e^{1/x} - 2x$

(b)

$$\begin{aligned}
\frac{d}{dx}\int_{x^3}^2 e^{t^2} - \sin(t)\,dt &= \frac{d}{dx}\left(-\int_2^{x^3} e^{t^2} - \sin(t)\,dt\right) \\
&= \frac{d}{dx}\int_2^{x^3} -e^{t^2} + \sin(t)\,dt \\
&= \left(-e^{x^6} + \sin(x^3)\right)\frac{d}{dx}\left[x^3\right] \\
&= \left(-e^{x^6} + \sin(x^3)\right)3x^2.
\end{aligned}$$

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5. Evaluate the following definite and indefinite integrals. (8 points)

(a) $\displaystyle\int_1^e \frac{2\ln x}{x}\,dx$

(b) $\displaystyle\int \tan x \cos^6 x\,dx$

SOLUTION.

(a) First, factor out the constant

$$2\int_1^e \frac{\ln x}{x}\,dx.$$

Second, for the integrant $\frac{\ln x}{x}$, substitute $u = \ln(x)$ and $du = \frac{1}{x}dx$, hence;

$$2\int_1^e \frac{\ln x}{x}\,dx = 2\int_{\ln(1)}^{\ln(e)} u\,du = u^2\Big]_0^1 = 1.$$

(b) First, notice that $\tan(x)\cos^6(x) = \cos^5(x)\sin(x)$. Second, for the integrant $\sin(x)\cos^5(x)$, substitute $u = \cos(x)$ and$du = -\sin(x)\,dx$, hence;

$$\begin{aligned}\int \tan x \cos^6 x\,dx &= \int \cos^5 \sin(x)\,dx \\ &= -\int u^5\,du \\ &= -\frac{u^6}{6} + C.\end{aligned}$$

Lastly, we substitute back for $u = \cos(x)$, thus,

$$\int \tan x \cos^6 x\,dx = -\frac{\cos^6(x)}{6} + C.$$

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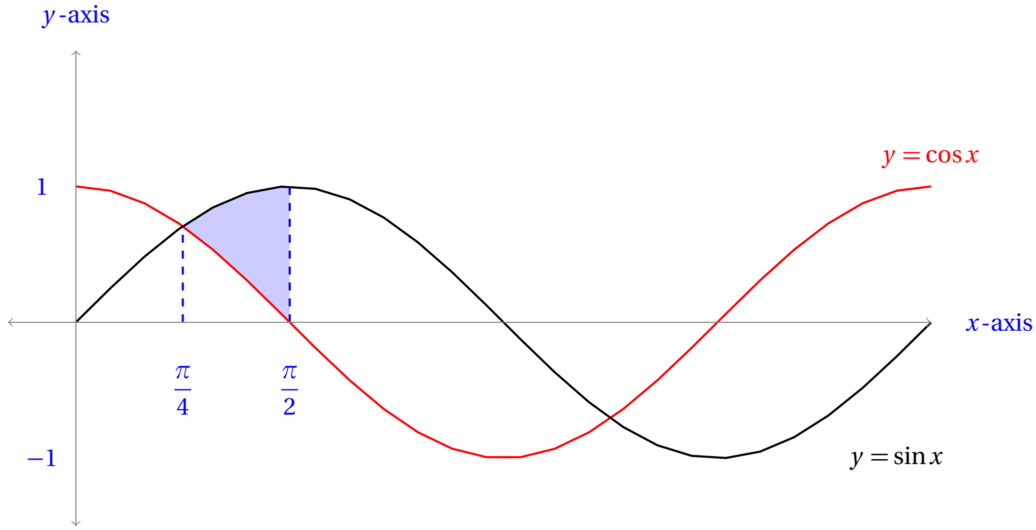
6. Set up the definite integral you would take to find each of the following volumes or areas. Do not evaluate. (12 points)

(a) The area of the shaded region, between the first intersection of $\sin x$ and $\cos x$ and $\pi/2$.

(b) The object obtained by rotating the region bounded by $y = \sqrt{x}$ and $y = \dfrac{1}{2}x$ about the line $x = -2$.

SOLUTION.

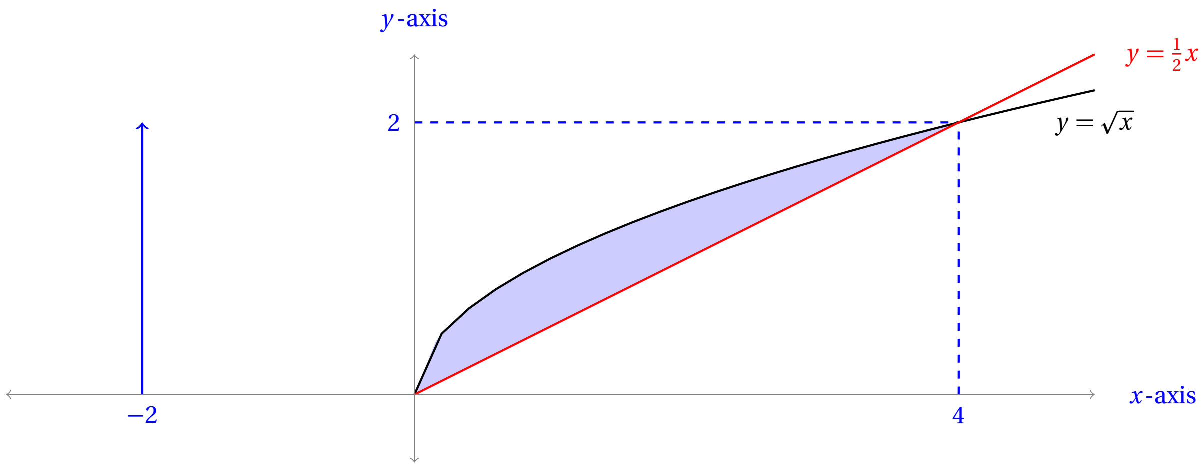
(a) The shaded region is given by



Now, an application of the Net Change Theorem reveals that the integral representing our desired area is given by

$$\int_{\pi/4}^{\pi/2} [\sin(x) - \cos(x)]\, dx = \sqrt{2} - 1$$

The shaded region of the area that we aim to rotate is given by



First, note that $y = \frac{1}{2}x$ if and only if $x = 2y$, and $y = \sqrt{x}$ implies that $x = y^2$. Second, recall that if a region is bounded by the curves $x = r(y), x = C, y = a$, and $y = b$ is rotated by the line $x = C$, then the corresponding volume of revolution is

$$\boxed{V = \int_a^b \pi \left[r(y) - C\right]^2 \, dy.}$$

Now, we proceed by means of **washer method**. That is, if V_1 is the volume obtained by rotating the region under $x = 2y$ from $y = 0$ to $y = 2$ around the line $x = -2$, and V_2 is the corresponding volume for the curve $x = y^2$, then the desired volume is given by

$$V = V_1 - V_2.$$

Applying our formula to calculate each of V_1 and V_2, we obtain

$$\begin{aligned}
V &= V_1 - V_2 \\
&= \int_0^2 \pi \left[r_1(y) + 2\right]^2 \, dy - \int_0^2 \pi \left[r_2(y) + 2\right] \, dy \\
&= \pi \int_0^2 \left[r_1(y) + 2\right]^2 - \left[r_2(y) + 2\right]^2 \, dy \\
&= \pi \int_0^2 \left[2y + 2\right]^2 - \left[y^2 + 2\right]^2 \, dy \\
&= \pi \int_0^2 \left[4y^2 + 8y + 4\right] - \left[y^4 + 4y^2 + 4\right] \, dy \\
&= \pi \int_0^2 \left(8y - y^4\right) \, dy \\
&= \frac{48\pi}{5}.
\end{aligned}$$

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7. Evaluate the following integrals by any technique. Be sure to show your work or give an explanation. (10 points)

(a) $\displaystyle\int \frac{x^5}{\sqrt{x^2-2}}\,dx$

(b) $\displaystyle\int_0^2 |x-1|\,dx.$

SOLUTION.

(a) For the integrant $\dfrac{x^5}{\sqrt{x^2-2}}$, substitute $u = x^2-2$ and$du = 2x\,dx$, hence;

$$\begin{aligned}\int \frac{x^5}{\sqrt{x^2-2}}\,dx &= \int \frac{(u+2)^2}{2\sqrt{u}}\,du \\ &= \int \frac{1}{2}u^{3/2} + 2u^{1/2} + 2u^{-1/2}\,du \\ &= \frac{1}{5}u^{5/2} + \frac{4}{3}u^{3/2} + 4u^{1/2} + C.\end{aligned}$$

Lastly, we substitute back for $u = x^2-2$, thus,

$$\int \frac{x^5}{\sqrt{x^2-2}}\,dx = \frac{1}{5}(x^2-2)^{5/2} + \frac{4}{3}(x^2-2)^{3/2} + 4(x^2-2)^{1/2} + C.$$

(b) Since

$$|x-1| = \begin{cases} -(x-1), & \text{when } x \le 1, \\ (x-1), & \text{when } x \ge 1, \end{cases}$$

we see that

$$\int_0^2 |x-1|\,dx = \int_0^1 -(x-1)\,dx + \int_1^2 (x-1)\,dx = \frac{1}{2} + \frac{1}{2} = 1.$$

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