(1) Consider the region $S$ bounded by the three curves 

$$y = \arccos x, \quad x = 0 \quad \text{and} \quad y = \frac{\pi}{3}$$

Calculate the area of $S$

**Answer:** One way of writing the required integral is \( \int_{0}^{\frac{1}{2}} \arccos x - \frac{\pi}{3} \, dx \). But we do not know how to do this integral.

The critical observation is that \( y = \arccos x \) is the same as saying \( x = \cos y \). So the area is

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos y \, dy = \sin y \big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \sin \frac{\pi}{2} - \sin \frac{\pi}{3} = 1 - \frac{\sqrt{3}}{2}. $$

(2) Using the cylindrical shell method, find the resulting volume if the region between the three curves \( y = \cos x \), \( y = 0 \) and \( x = \frac{\pi}{3} \) is rotated around the $y$-axis.

**Answer:** The region described is exactly the region of problem 1, with the roles of $x$ and $y$ reversed.

The cylinder through the value $x$ has height $\cos x$ so it has area

$$A(x) = 2\pi x \cos x.$$ 

It follows that the volume of the resulting solid is

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} A(x) \, dx = 2\pi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} x \cos x \, dx.$$ 

This can be integrated by parts, with $u = x$ and $v = \sin x$. Then

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x.$$ 

So the volume is

$$2\pi (x \sin x + \cos x) \big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} = 2\pi \left( \frac{\pi}{2} - \frac{\pi}{3} \frac{\sqrt{3}}{2} - \frac{1}{2} \right) = \pi^2 \left( 1 - \frac{1}{\sqrt{3}} \right) - \pi.$$
(3) A reluctant burro is pulled along a path by a man who must exert a force of

\[ \frac{10}{(1 + x)^2} \]

pounds when the burro is a distance x feet from the beginning of the path. How much work does he need to do to move the burro 4 feet down the path?

**Answer:** The work

\[ W = \int F(x)dx = 10 \int_0^4 \frac{dx}{(1 + x)^2}. \]

If we make the substitution \( u = x + 1 \Rightarrow dx = du \) we get

\[ 10 \int_{u=1}^{u=5} \frac{du}{u^2} = -10 \left[ \frac{1}{u} \right]_1^5 = 10(1 - 1/5) = 8 \text{ ft-lbs}. \]

(4) On the planet PsK! the standard unit of length is the gronka, abbreviated gr. Acceleration due to gravity is always 20gr/sec² downwards. A ball is dropped from the top of a very tall tower.

- What will the velocity of the ball be after \( t \) seconds?
  **Answer:** Acceleration is the derivative of velocity, so \( v = 20t + C \) downwards. When \( t = 0 \) the velocity is 0, so \( C = 0 \). Hence \( v = 20t \) gr/sec downwards.

- How far will the ball have dropped after \( t \) seconds?
  **Answer:** Velocity is the derivative of the distance dropped, so \( r(t) = \int 20tdt = 10t^2 + C \). When \( t = 0 \) the ball has not dropped at all, so \( C = 0 \). Hence \( r = 10t^2 \) gr.

- What will the velocity be when the ball has dropped \( r \) gronkas?
  **Answer:** Since \( r = 10t^2, t = \sqrt{r/10} \). Then \( v = 20t = 20\sqrt{r/10} = \frac{2\sqrt{10}r}{r} \)

- What is the average velocity of the ball over the first 5 gronkas?
  **Answer:** Since \( v = 2\sqrt{10}r \), the average will be \( \frac{\int_0^5 2\sqrt{10}rdt}{5} = \frac{\frac{2\sqrt{10}}{5}r^{3/2}}{5} \]

\[ = \frac{4}{5}\sqrt{50} = \frac{20\sqrt{2}}{5} \text{ gr/sec}. \]

(5) The huge City University of Elbonia admits anyone with a Math SAT of 400 or over. Here is a graph of the number of students admitted for each SAT score between 400 and 800 (a perfect score). For example, according to the graph, about 57 students had a Math SAT of 635. Using \( n = 4 \) on the interval \([400, 800]\) estimate the total number of students admitted.

**Answer:** In this case \( \Delta x = (800 - 400)/4 = 100, x_0 = 400, x_1 = 500, x_2 = 600, x_3 = 700, x_4 = 800 \). According to the graph, \( f(x_0) = 30, f(x_1) = 50, f(x_2) = 60, f(x_3) = 40, f(x_4) = 20 \).
For the Trapezoidal rule: \( f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4) = 30 + 100 + 120 + 80 + 20 = 350 \) so the estimate is \( 350 \frac{\Delta x}{2} = 350 \cdot 50 = 17,500 \).

For Simpson’s rule: \( f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4) = 30 + 200 + 120 + 160 + 20 = 530 \) so the estimate is \( 530 \frac{\Delta x}{3} \approx 530 \cdot 33 \frac{1}{3} = 17,667 \).