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<th>Question #</th>
<th>Score</th>
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<td>1</td>
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<td>Bonus</td>
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<td>3</td>
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</tbody>
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Total: /20
1. [5 marks] Using Green's Theorem, calculate $\int_C \mathbf{F} \cdot d\mathbf{s}$, where

$$\mathbf{F}(x, y) = (xy + \cosh(y), x \sinh(y))$$

and $C$ is the boundary (with the positive orientation) of the triangle enclosed by the lines $y = 3x$, $y = x$ and $x = 1$.

*Recall that* $\cosh(y) = \frac{1}{2}(e^y + e^{-y})$ *and* $\sinh(y) = \frac{1}{2}(e^y - e^{-y})$.

By Green's Theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D \left( \sinh(y) - (x + \sinh(y)) \right) dy \, dx$$

$$= \int_0^3 \int_0^{3x} \left( \sinh(y) - (x + \sinh(y)) \right) dy \, dx$$

$$= \int_0^3 \left[ \frac{1}{2} (e^y - e^{-y}) - x - x \sinh(y) \right]_0^{3x} dx$$

$$= \int_0^3 \left( -2x^2 \right) dx$$

$$= \frac{-2}{3}$$
2. [5 marks] Let \( W \) be the region that consists of a box determined by the vectors \( 2i, 2j \) and \( 3k \), with a unit cube cut out of the corner. Let

\[
F(x, y, z) = (2yz + x^2, 2xy + z^2, 2xz + y^2). 
\]

Calculate the flux of \( F \) across the surface of \( W \).

If \( S_1 \) is the surface of the big box, and \( S_2 \) is the surface of the little box then we have:

\[
\iint_{S \cap W} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} - \iint_{S_2} \vec{F} \cdot d\vec{S}
\]

(This is true since \(-\iint_{S_2} \vec{F} \cdot d\vec{S}\) calculates the flux into the little box, which is out of \( S \)).

By the Divergence Theorem,

\[
\iiint_{\text{big box}} \vec{F} \cdot \nabla dV - \iiint_{\text{little box}} \vec{F} \cdot \nabla dV = \iiint_{W} \nabla \cdot \vec{F} dV
\]

\[
= \iiint_{W} 2x + 2y + 2z dV
\]

\[
= \iiint_{W} 6x dV - \iiint_{W} \frac{1}{2} \frac{1}{2} \frac{1}{2} dV
\]

\[
= \left( \int_{0}^{2} \int_{0}^{2} \int_{0}^{2} 6x dxdydz \right) - \left( \int_{0}^{2} \int_{0}^{2} \int_{0}^{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} dxdydz \right)
\]

\[
= \left( \frac{2}{3} \cdot 2 \cdot 2 \cdot 2 \right) - \left( \frac{1}{3} \cdot 2 \cdot 2 \cdot 2 \right)
\]

\[
= 6 \cdot 3 \cdot 2 - 1 \cdot 3 \cdot 2
\]

\[
= 6 \cdot 9
\]

\[
= 54
\]
3. [5 marks] Let $S$ be a surface to which Stoke's Theorem can be applied. Explain a strategy you might employ to calculate the surface area of $S$ (i.e., $\int_S dS$) using a line integral.

Stoke's Theorem says that if $\vec{\gamma}$ is the boundary of $S$, then

$$\int_{\vec{\gamma}} \vec{F} \cdot d\vec{S} = \iint_S \text{curl}(\vec{F}) \cdot \hat{n} \, dS$$

$$= \iint_S \left[ \text{curl}(\vec{F}) \cdot \frac{\vec{N}}{||\vec{N}||} \right] dS$$

Hence, if $\vec{F}$ is a vector field such that $\text{curl}(\vec{F}) \cdot \frac{\vec{N}}{||\vec{N}||} = 1$ we have that $\int_{\vec{\gamma}} \vec{F} \cdot d\vec{S}$ equals the surface area of $S$.

For example, if $\vec{F}$ is such that $\text{curl}(\vec{F}) = \frac{\vec{N}}{||\vec{N}||}$, then

$$\int_{\vec{\gamma}} \vec{F} \cdot d\vec{S} = \text{Surface Area}. \quad (\text{But such an } \vec{F} \text{ may not exist!})$$

4. [5 marks]

Consider the two sequences $\{a_n\}_{n=1}^{\infty}$ where $a_n = \frac{n(2n+1)}{n^2}$, and $\{b_n\}_{n=1}^{\infty}$ where $b_n = \frac{(-1)^n}{5}$. For the following list of sequences, indicate if they are convergent or divergent, and find the limits of those that do converge (You don't need to justify your choices).

(a) $\{a_n\}_{n=5}^{\infty}$ Convergent to 2

(b) $\{b_n\}_{n=1}^{\infty}$ Divergent $\frac{1}{2}$

(c) $\{3a_n - 1\}_{n=2}^{\infty}$ Convergent to 5

(d) $\{b_n^2\}_{n=1}^{\infty}$ Convergent to $\frac{1}{25}$

(e) $\{1/a_n\}_{n=1}^{\infty}$ Convergent to $\frac{1}{2}$

(f) $\{a_n/b_n\}_{n=1}^{\infty}$ Divergent $\frac{1}{2}$
Bonos: Prove (using the definition of a convergent sequence) that the sequence \((1, 0, 1, 0, 1, 0, \ldots)\) diverges.

A sequence converges (to a limit \(L\)) if for any \(\varepsilon > 0\), there exists an \(N\) such that

\[L - \varepsilon < a_n < L + \varepsilon\]

for all \(n \geq N\).

So to show \((1, 0, 1, 0, \ldots)\) diverges, we need to show that there exists an \(\varepsilon > 0\) such that no matter what the values of \(L\) and \(N\) are, we can always find an \(n \geq N\) with \(a_n \notin (L - \varepsilon, L + \varepsilon)\).

Take \(\varepsilon = \frac{1}{4}\).

Then either \(0 < L - \varepsilon\) or \(L + \varepsilon < 1\).

So in either case we can find an \(n \geq N\) with \(a_n \notin (L - \varepsilon, L + \varepsilon)\).

So the sequence is indeed divergent.
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/20
1. [7.5 marks (1.5 each)] Determine if the following series converge absolutely, converge conditionally or diverge. Be sure to justify your answers!

(a)

\[
\sum_{k=1}^{\infty} \frac{2}{4 + 2^{-k}}
\]

Since

\[
\lim_{k \to \infty} \frac{2}{4 + \frac{1}{2^k}} = \frac{1}{2} \neq 0,
\]

the series diverges.

(b)

\[
\sum_{k=0}^{\infty} \frac{(-2)^k}{3^{k+1}}
\]

\[
= \frac{1}{3} \sum_{k=0}^{\infty} \left(-\frac{2}{3}\right)^k
\]

So series converges (geometric series and \(-1 < -\frac{2}{3} < 1\)).
(c) \[ \sum_{k=2}^{\infty} \frac{(-1)^k}{\sqrt{k(k-1)}} \]

\[ \left| \frac{(-1)^k}{\sqrt{k(k-1)}} \right| > \frac{1}{\sqrt{(k-1)^2}} = \frac{1}{k-1} \]

But \[ \sum_{k=2}^{\infty} \frac{1}{k-1} \] diverges (it's a Harmonic series)

so by the comparison test, so does \[ \sum_{k=2}^{\infty} \left| \frac{(-1)^k}{\sqrt{k(k-1)}} \right| \]

But it is an alternating series, \( \frac{1}{\sqrt{k(k-1)}} \to 0 \) \& \( k \),

and \( \frac{1}{\sqrt{k(k-1)}} \to 0 \). So by Alt. series test, the series converges.

Hence the series is conditionally convergent.

(d) \[ \sum_{k=1}^{\infty} (-1)^{k+1} e^{-k} \]

Converges by, e.g. Geometric series test \( \left( r = \frac{-1}{e} \right) \)

- Root test
- Ratio test absolutely
(e) \[ \lim_{{k \to \infty}} \frac{8^{k+1}}{(2k+2)!} = \lim_{{k \to \infty}} \frac{8^k}{(2k)!} = \lim_{{k \to \infty}} \frac{8}{(2k)(2k+1)} = 0. \]

So series converges.

2. [4+1.5 marks]

(a) Let \( \cosh(x) = (e^x + e^{-x})/2 \). Find a formula for the \( n \)th Taylor coefficient (about 0), then write the Taylor series in \( \Sigma \)-notation.

\[
\cosh(x)' = \frac{e^x - e^{-x}}{2} \quad \text{so obviously} \quad \cosh(x) = \begin{cases} (\cosh(x)), & \text{even} \\ (\sinh(x)), & \text{odd} \end{cases}
\]

\[
\cosh(x)'' = \frac{e^x + e^{-x}}{2}
\]

\[
\therefore \cosh(x)^{(n)} = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}
\]

\[
\therefore \cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}
\]
(b) Suppose you only cared about $x \in [0, 4]$. Describe how you would determine an $n$ so that $n$th Taylor polynomial (about 0) is within $10^{-6}$ of the actual value of $\cosh(x)$. You don't need to find $n$, just describe (with as much detail as possible) the calculation you would make.

If $x \in [0, 4]$ then clearly $\cosh(x)^{(n)} \leq \frac{e^{x} + e^{-x}}{2}$

So to find $n$ we need to find it so that $\left(\frac{e^{x} + e^{-x}}{2}\right) \cdot 4^{n+1} \leq 10^{-6}$

3. [3 marks] Find the radius of convergence for the series

$$\sum_{k=2}^{\infty} \frac{x^k}{\ln(k)}$$

Ratio test: $\lim_{k \to \infty} \left| \frac{x^{k+1}}{x^k \frac{\ln(k+1)}{\ln(k)}} \right| = \lim_{k \to \infty} \left| x \cdot \frac{\ln(k)}{\ln(k+1)} \right| = |x| \cdot \lim_{k \to \infty} \frac{\ln(k)}{\ln(k+1)} = |x| \cdot 1$ (by L'Hôpital's rule)

Now $|x| < 1$ precisely when $|x| < 1$

i. radius of convergence is 1
4. [4 marks] Calculate the first 4 terms of the Taylor series about 0 of \( f'(x) \) where \( f(x) = \frac{\sin(x)}{e^x} \).

\[
\frac{f(x)}{x^n} = \frac{O + x + O x^2 - \frac{1}{3!} x^3 + O x^4}{1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4}
\]

Suppose \( f(x) = \sum_{n=0}^{\infty} a_n x^n \). Then

\[
a_0 b_0 = c_0 \Rightarrow a_0 \cdot 1 = 0 \Rightarrow a_0 = 0
\]

\[
a_0 b_1 + a_1 b_0 = c_1 \Rightarrow 0 \cdot 1 + a_1 \cdot 1 = 1 \Rightarrow a_1 = 1
\]

\[
a_0 b_2 + a_1 b_1 + a_2 b_0 = c_2 \Rightarrow 0 \cdot \frac{1}{2} + 1 \cdot 1 + a_2 \cdot 1 = 0 \Rightarrow a_2 = -1
\]

\[
a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 = c_3 \Rightarrow 0 \cdot \frac{1}{3!} + 1 \cdot \frac{1}{2!} + (-1) \cdot 1 + a_3 \cdot 1 = -\frac{1}{3!} \Rightarrow a_3 = \frac{1}{3}
\]

\[
a_0 b_4 + a_1 b_3 + a_2 b_2 + a_3 b_1 + a_4 b_0 = c_4 \Rightarrow 0 \cdot \frac{1}{4!} + 1 \cdot \left(\frac{1}{3!}\right) + (-1) \cdot \left(\frac{1}{2!}\right) + \frac{1}{3} \cdot 1 + a_4 = 0
\]

\[
\Rightarrow a_4 = \frac{1}{2} - \frac{1}{6} - \frac{1}{3} = 0
\]

So \( f(x) = x - x^2 + \frac{1}{3} x^3 + O x^4 + \ldots \)

\[\therefore f'(x) = 1 - 2x + x^2 + O x^3 + \ldots\]
(Bonus) Let $p$ be a fixed positive integer. Show the power series

$$\sum_{k=0}^{\infty} \frac{(pk)!}{(k!)^p} x^k$$

has radius of convergence $1/p^p$.

Ratio:

$$\lim_{k \to \infty} \left| \frac{(p(k+1))!}{(k!p)^p} x^{k+1} \right| \left| \frac{(pk)!}{(k!)^p} x^k \right|$$

$$= \lim_{k \to \infty} |x| \left( \frac{(p(k+1))!}{(pk)!} \cdot \frac{K!^p}{(k!)^p} \right)$$

$$= \lim_{k \to \infty} |x| \left[ (p+1)(p+2) \cdots (p+1) \cdot \left( \frac{1}{K+1} \right)^p \right]$$

$$= \lim_{k \to \infty} |x| \left[ p^p \left( \frac{(k+1)(k+1-p) \cdots (k+1-p)}{(k+1)(k+1-p) \cdots (k+1)} \right)^p \right]$$

$$= \lim_{k \to \infty} |x| \cdot p^p \left( \frac{(k+1)(k+1-p) \cdots (k+1-p)}{(k+1)(k+1-p) \cdots (k+1)} \right)^p$$

fraction of polynomials
of degree $p$, with coefficients
of $k^p$ both equal to 1.

So $|x| \cdot p^p < 1$ precisely when $|x| < \frac{1}{p^p}$.

So radius of convergence is $\frac{1}{p^p}$.