Differential Equations Reference Sheet

**Basic Equations**

For an equation of the form,

\[ f' = h \]

antidifferentiate both sides to obtain

\[ f = H + C \]

where \( H' = h \) and \( C \) is the constant of integration. Similarly, for

\[ f^{(n)} = h \]

antidifferentiate both sides \( n \) times to get

\[ f = H + C_n x^{n-1} + \cdots + C_2 x + C_1 \]

\[ f(y) = H(y) + C_n y^{n-1} + \cdots + C_2 y + C_1 \]

where \( H^{(n)} = h \).

**Separable Equations**

For an expression of the form,

\[ f' (g \circ f) = h \]

rewrite the left-hand as

\[ (G \circ f)' = h \]

where \( G' = g \), and antidifferentiate both sides.

**The Method of Integrating Factors**

Given,

\[ f' + g f = h \]

multiply through by an integrating factor, \( \exp \circ G \), where \( G' = g \),

\[ (\exp \circ G) \cdot f' + (\exp \circ G) \cdot g f = (\exp \circ G) \cdot h \]

\[ e^{G(s)} \frac{df}{ds} + e^{G(s)} g(s)f(s) = e^{G(s)} h(s) \]

rewrite the left-hand side as

\[ [(\exp \circ G) \cdot f]' = (\exp \circ G) \cdot h \]

\[ \frac{d}{ds} e^{G(s)} f(s) = e^{G(s)} h(s) \]

and antidifferentiate.
A Special Trick for Separable Equations

If you use the classical notation with the function \( y : \mathbb{R} \to \mathbb{R} \) and suppress the auxiliary symbol "\( (x) \)", then you can use the special notation trick,

\[
g(y) \frac{dy}{dx} = h(x) \quad \implies \quad g(y) \, dy = h(x) \, dx \quad \implies \quad \int g(y) \, dy = \int h(x) \, dx
\]

where we define

\[
g(y) = g \circ y \quad \text{(definition)}
\]

and

\[
\int g(y) \, dy = G \circ y + C \quad \text{(definition)}
\]

with \( G' = g \).

Antidifferentiation Rules

Corresponding to the three rules of differentiation and integration, there are three rules of antidifferentiation (also called indefinite integration).

\[
\begin{align*}
\int (f + g) &= \int f + \int g \\
\int fg &= \int Fg - \int Fg' \\
\int g'(f \circ g) &= F \circ g + C
\end{align*}
\]

\[
\begin{align*}
\int f(x) + g(x) \, dx &= \int f(x) \, dx + \int g(x) \, dx \\
\int f(y)g(y) \, dy &= F(y)g(y) - \int F(y) \frac{dg}{dy} \, dy \\
\int f(g(t)) \frac{dg}{dt} \, dt &= F(g(t)) + C
\end{align*}
\]

where \( F' = f \).