Worksheet 4
The Exponential Map

Recall that
\[ e^t := 1 + t + \frac{1}{2!} t^2 + \frac{1}{3!} t^3 + \frac{1}{4!} t^4 + \cdots \]
\[ \cos(t) := 1 - \frac{1}{2!} t^2 + \frac{1}{4!} t^4 - \frac{1}{6!} t^6 + \cdots \]
\[ \sin(t) := t - \frac{1}{3!} t^3 + \frac{1}{5!} t^5 - \frac{1}{7!} t^7 + \cdots \]

1. Prove Euler’s formula:
\[ e^{it} = \cos(t) + i \sin(t) \]

Notice that, by taking \( t = \pi \), we obtain the famous expression
\[ e^{i\pi} + 1 = 0 \]

Recall that the zero map \( 0 \in \text{Lin}(V) \) and identity map \( I \in \text{Lin}(V) \) are given by
\[ 0(x) = 0 \]
\[ I(x) = x \]

for all \( x \in V \). Since
\[ I^2(x) := I(I(x)) = I(x) = x \]

for any \( x \in V \), it follows that \( I^2 \) is the identity map. That is, \( I^2 = I \).

2. What is \( I^{500} \)? \( 0^{27} \)?

3. Compute \( e^{i0} \).

4. Compute \( e^{it} \). (Hint. Factor out an \( I \) and use the definition of the function \( e^t \) given above.)

5. Show that the matrix
\[ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

corresponds to the zero transformation on \( \mathbb{R}^2 \). That is, you must show that it sends every column vector to the zero vector \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{R}^2 \).

6. Show that the matrix
\[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

corresponds to the identity transformation on \( \mathbb{R}^2 \).
Define

\[ Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]

7. Find an expression for

\[ e^{tQ} := I + tQ + \frac{1}{2!} t^2 Q^2 + \frac{1}{3!} t^3 Q^3 + \frac{1}{4!} t^4 Q^4 + \cdots \]

in terms of \( \sin(t) \) and \( \cos(t) \).  

(Hint. Find a pattern in the powers \( Q^k \) of \( Q \), then use the definitions of 
\( \text{sine and cosine.} \))

Let \( a, b \in \mathbb{R} \) be real numbers, and let

\[ M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \]

and

\[ N = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \]

8. Compute the exponential \( e^{tM} \).

9. (Slightly more difficult.) Compute \( e^{tN} \).

Let \( A, B \in \text{Lin}(V) \) and suppose that \( B \) has an inverse \( B^{-1} \in \text{Lin}(V) \),

\[ BB^{-1} = B^{-1}B = I \]

9. Show that

\[ e^{B^{-1}AB} = B^{-1}e^A B \]

(Hint. Notice that \( (B^{-1}AB)^2 = B^{-1}AB B^{-1}AB = B^{-1}A^2B \). What is \( (B^{-1}AB)^{200} \) ?)