Math 6A Final Exam

Name:  

Circle the correct answer.  

1. What is the waggle dance?
   (a) a dance that bees do when they hear music
   (b) the first higher kinematic pair
   (c) a dance that bees do to communicate the amount and location of food
   (d) a robot planning problem

2. Which topic is used in the construction of the Jerusalem Chords bridge and in the design of fonts?
   (a) classical mechanics
   (b) Bézier curves
   (c) the right-hand rule
   (d) anisotropic materials

3. What is a common application of knot theory to biology?
   (a) studying the flight patterns of honeybees
   (b) the right-hand rule
   (c) untangling snakes (also eels)
   (d) inhibiting the unwinding of DNA in viruses
4. Identify the Sierpinski triangle.

4
1 1
1 2 1
1 3 3 1

(a) 1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

(b) ![Image of Sierpinski Triangle]

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5. What is a common hypothetical task in the field of robot motion planning?

(a) the dance competition problem
(b) the piano mover’s problem
(c) defeating the humans
(d) teaching robots to teach other robots to solve the problem instead

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2
Let $T$ be the torus in $\mathbb{R}^3$ consisting of all points that are distance 1 from the circle $x^2 + y^2 = 4$, $z = 2$, and let $n$ be the outward unit normal vector field on $T$.

6. Compute

$$\int_S \partial_y \cdot n \, dS$$

where $S$ is the portion of $T$ which lies in the first octant.$^1 \quad (5 \text{ pts})$

$$S = S \cup O_1 \cup O_2 \quad \leftarrow \text{closed surface}$$

$$\int_S \partial_y \cdot n \, dS = \int_S \partial_y \cdot n \, dS - \int_{O_1} \partial_y \cdot n \, dS - \int_{O_2} \partial_y \cdot n \, dS$$

$$= \int_{O_1} (\partial_y \cdot \mathbf{n}) \, dV + \int_{O_2} \, dS$$

$$= \text{area} (O_1) = \pi$$

$^1$Recall that the first octant is the region of $\mathbb{R}^3$ given by $x, y, z \geq 0$. 
7. What is the total amount of charge $Q$ in the region $x^2 + y^2 + z^2 \leq 1$, $y \geq 0$, when the electrostatic field is $E = \rho^2 \hat{\rho}$? \footnote{Recall Gauss' law: The total charge enclosed by a surface in a vacuum is equal to $\varepsilon_0$ times the flux of the electrostatic field through that surface. You might also want to recall that the surface area of a sphere is equal to $4\pi$ times the square of the length of the radius.} Here $\rho$ denotes the spherical coordinate function $\rho$. \hspace{1cm} (5 pts)

\[ Q = \varepsilon_0 \int_S E \cdot n \, dS \]

\[ = \varepsilon_0 \int_{\text{top}} (\hat{\rho} \cdot \nabla) \, dS + \varepsilon_0 \int_{\text{bottom}} 0 \, dS \]

\[ = \varepsilon_0 \text{ area (top)} \]

\[ = \varepsilon_0 \left( \frac{1}{2} \cdot 4\pi \right) \]

\[ = 2\pi \varepsilon_0 \]
Let $S$ be the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$. Equip $S$ with the orientation induced by the unit normal vector $n = \partial_z$ at the point $(0,0,2) \in S$. In other words, $(\partial_x, \partial_y)$ is a positively-oriented basis of the tangent space of $S$ at $(0,0,2)$.

8. What is the electric current $I$ through $S$ when the magnetic field $B = \cos^2(z) \partial_\theta$? $^3$ (5 pts)

\[ I = \frac{1}{\mu_0} \int_S \left( \text{curl} \ B \right) \cdot n \, dS \]
\[ = \frac{1}{\mu_0} \int_{\partial S} B \cdot d\ell \]
\[ = \frac{1}{\mu_0} \int_0^{2\pi} \left( \partial_\phi \left( \partial_\phi \right) \right) \, dt \]
\[ = \frac{8\pi}{\mu_0} \]

$^3$ Hint. Recall Ampère's law: $I = \frac{1}{\mu_0} \int_S (\text{curl} \ B) \cdot n \, dS$. You might also want to use the fact that the magnitude of $\partial_\theta$ at a point $p \in \mathbb{R}^3$ is equal to the distance from $p$ to the $z$-axis.
Extra Credit Questions. (1 pt ea.)

(i) What was Ostrogradsky’s original interest in the divergence theorem?

The study of heat.

(ii) Who first introduced the idea of the dot product and the cross product?

Gibbs

(iii) Why does Wigner consider the effectiveness of mathematics in the natural sciences to be “unreasonable”?

Mathematics is motivated by the pursuit of beauty, while the sciences are driven by the quest for knowledge, and there is no a priori reason to believe the two to be related.

(Answers may vary.)