I am interested in symplectic geometry, moduli spaces, geometric quantization, and related fields. More generally, I am interested in interactions between physics and mathematics. The following sections are arranged so the most recent material appears first.

1 Polysymplectic Geometry

In [3] and [4], I introduced a theory of vector-valued symplectic geometry. This independently developed theory is essentially equivalent to the polysymplectic formalism of Günther [8], the k-symplectic formalism of Awane [2], the generalized symplectic geometry of Norris [15], and the theory of p-almost cotangent structures of de León, Méndez, and Salgado [7].

**V-symplectic manifolds**

Consider a manifold $M$ and vector space $V$.

**Definition 3.1** ([4]). A $V$-symplectic structure on $M$ is a closed $V$-valued 2-form $\omega \in \Omega^2(M,V)$ which is nondegenerate in the sense that $i_X \omega = 0$ only if $X = 0$.

Examples include,

1. The semisimple Lie group $G$ with $\mathfrak{g}$-symplectic structure $-d\theta$, and $\theta \in \Omega^1(G,\mathfrak{g})$ is the Maurer-Cartan form on $G$. The model space is $\mathfrak{g}$, with linear $\mathfrak{g}$-symplectic structure given by the Lie bracket.

2. The phase space $\text{Hom}(TQ,V)$ with $V$-symplectic structure $-d\theta$, and $\theta$ is given by $\theta_g(X) = \phi(\pi_*X)$. The model space is $U \oplus \text{Hom}(U,V)$, with linear $V$-symplectic structure $\omega(u + \phi, u' + \phi') = \phi'(u) - \phi(u')$, where $U$ is the model space of $Q$.

The second example is distinguished by the following property.

**Theorem 3.6** ([4]). Every $V$-symplectic manifold $(M,\omega)$ locally polysymplectically embeds in $\text{Hom}(TM,V)$.

If $\omega = -d\theta$ is exact, then $\theta : M \to \text{Hom}(TQ,V)$ provides a polysymplectic embedding. In general, $\theta$ exists only locally. It is interesting to observe that certain longstanding open problems in symplectic geometry may be quickly settled in the polysymplectic case, for example, the following.

**Arnold Conjecture** (See [13], Chapter 11). A symplectomorphism that is generated by a time-dependent Hamiltonian vector field should have at least as many fixed points as a Morse function on the manifold must have critical points.

**Theorem 3.21** ([4]). The Arnold conjecture fails in the $V$-symplectic setting.

A counterexample is provided by left multiplication on the $\mathfrak{g}$-symplectic manifold $(G, -d\theta)$. The fundamental theorem of $V$-Hamiltonian manifolds is the following.

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Theorem 3.22 ([4]). Let \((M, \omega, G, \mu)\) be a \(V\)-Hamiltonian systems and fix \(\alpha \in \text{Hom}(\mathfrak{g}, V)\). If the stabilizer subgroup \(G_\alpha\) of \(\alpha\) under the coadjoint action is connected, and if \(M_\alpha = \mu^{-1}(\alpha)/G_\alpha\) is smooth, then there is a unique \(V\)-valued 2-form \(\omega_\alpha \in \Omega^2(M_\alpha, V)\) such that
\[
\pi^* \omega_\alpha = i^* \omega,
\]
where \(i : \mu^{-1}(\alpha) \hookrightarrow M\) is the inclusion and \(\pi : \mu^{-1}(\alpha) \rightarrow M_\alpha\) is the projection. The form \(\omega_\alpha\) is closed and nondegenerate at \(\pi x\) if and only if \(\mathfrak{g}_{\pi x} = \mathfrak{g}_{\omega} \cap \mathfrak{g}_{x}\).

Unlike the symplectic case, the reduced \(V\)-valued 2-form may be degenerate. It should be noted that this result appeared earlier in [11].

Interactions with gauge theory

The aim of [4] is to exhibit the moduli space of flat connections over a manifold of arbitrary dimension as the polysymplectic reduction of the space of all connections. This extends an observation of Atiyah and Bott [1] in the case that \(M\) is a closed orientable surface.

Theorem 4.12 ([4]). Let \(M\) be a compact manifold of dimension at least 3, \(G\) a compact matrix Lie group, \(P\) a \(G\)-principal bundle on \(M\) with connected gauge group \(G\), \(A\) the space of connections on \(P\), and \(k > \frac{1}{2} \dim M + 1\) a fixed integer. Denote the the \(W^{k,2}\) Sobolev completion of \(A\) by \(\mathcal{A}_k\), and likewise for \(\mathcal{G}, g, \text{ and } \Omega^*,\) and write \(\widetilde{\Omega}^2(M)\) and \(\tilde{B}^2(M)\) for the spaces of \(C^1\) forms and coboundaries on \(M\), respectively. Let \(F : \mathcal{A}_k \rightarrow \Omega^2_{k-1}(M, \text{ad}P)\) be the curvature. The function
\[
\mu : \mathcal{A}_k \rightarrow \text{Hom}(\mathfrak{g}_{k+1}, \widetilde{\Omega}^2(M)/\tilde{B}^2(M)),
\]
given by
\[
\mu(A)(f) = (F_A \wedge f)_{\widetilde{\Omega}^2/\tilde{B}^2}, \quad f \in \Omega^0_{k+1}(M, \text{ad}P) \cong \mathfrak{g}_{k+1},
\]
is a moment map for the action of \(\mathcal{G}_{k+1}\) on \(\mathcal{A}_k\) with respect to the polysymplectic structure \(\omega \in \Omega^2(A, \widetilde{\Omega}^2(M)/\tilde{B}^2(M))\), defined by
\[
\omega(\alpha, \beta) = (\alpha \wedge \beta)_{\widetilde{\Omega}^2/\tilde{B}^2},
\]
for \(\alpha, \beta \in \Omega^1_{k}(M, \text{ad}P) \cong T_A \mathcal{A}_k\). The reduced space at 0 coincides with the moduli space of flat connections \(M_k = F^{-1}(0)/\mathcal{G}_{k+1}\) on \(P\). On the smooth points of \(M_k\), the reduced 2-form \(\omega_0\) takes values in the second cohomology \(H^2(M)\).

I obtain a similar characterization for the space of generalized Ricci flat connections on a holomorphic vector bundle \(E\) over a complex manifold \(M\).

Definition 4.21 ([4]). We call the connection \(A \in \mathcal{A}(E)\) Ricci flat if \(\text{tr } F_A = 0\).

Corollary 4.22 ([4]). Let \(M\) be a compact complex manifold and let \(E\) be a holomorphic vector bundle over \(M\) with \(c_1(E) = 0\). The moduli space of Ricci flat connections is the polysymplectic reduction of the space of connections \(\mathcal{A}_k(E)\).

2 The Volume of the Moduli Space of Flat Connections

The primary aim of [3] is to compute the volume of the moduli space of flat connections over manifolds of arbitrary dimension. In general, the volume of a symplectic manifold \((M^{2n}, \omega)\) is defined to be
\[
\text{vol } M = \int_M \omega^n/n!.
\]
Up to rescaling, \(\omega^n\) is the unique measure on \(M\) that is preserved by the group of symplectic transformations. The factor of \(1/n!\) may be motivated by the observation that \(\omega^n/n!\) is identified with the usual volume form on \(\mathbb{R}^{2n}\) by any symplectic coordinate chart.
Background: symplectic volume and the space of quantum states

Suppose \((L, \nabla)\) is a positive prequantum line bundle over the Kähler manifold \((M^{2n}, \omega)\). By this we will mean that \(L\) is a positive Hermitian line bundle on \(M\) with connection \(\nabla\) and curvature \(2\pi \omega \in \Omega^2(M, \mathbb{C}) \cong \Omega^2(M, \text{End} L)\). For each \(k > 0\), the \(k\)th tensor power \((L^k, k\nabla)\) is a prequantum line bundle for the rescaled symplectic manifold \((M, k\omega)\). The curvature condition ensures that \(L^k\) is holomorphic and the Riemann-Roch formula provides that

\[
\sum_{i=0}^{2n} (-1)^i \dim H^i(M, L^k) = \int_M \text{ch} L^k \wedge \text{Td} M.
\]

On the left-hand side, the Kodaira vanishing theorem yields \(H^i(M, L^k) = 0\) for \(i > 0\) and \(k \gg 0\). On the right-hand side, we have \(\text{ch} L^k = (k\omega)^n/n! + O(k^{n-1})\) and \(\text{Td} M = 1 + \Omega^2(M)\). Thus, in the large \(k\) limit,

\[
\dim H^0(M, L^k) = k^n \text{vol} M + O(k^{n-1}).
\]

In sum, the symplectic volume describes the growth rate of the quantum state space \(H^0(M, L^k)\) in the semiclassical limit \(h = \frac{2\pi}{k} \to 0\).

The volume of the moduli space

The moduli space of flat connections over a closed surface \(\Sigma\) is naturally a symplectic manifold. It arises, for example, as the space of dynamical solutions in classical Chern-Simons theory. In this situation, \(\Sigma\) represents a spacelike slice of a 3-dimensional spacetime. By the preceding discussion, the volume of the moduli space describes the growth rate of the quantum state space as the energy of the system tends to infinity.


In my PhD thesis [3], I determine the volume of the moduli space \(\mathcal{M}_G(M)\) over a base manifold \(M\) of arbitrary dimension, under special conditions. First, for a general base manifold \(M\) and a restricted structure group \(G\).

**Theorem 10.1** ([3]). Let \(M\) be a symplectic (resp. Riemannian) manifold and let \(T\) be a compact abelian Lie group equipped with an invariant metric. Then

\[
\text{vol} \mathcal{M}_T(M) = \text{vol} (T)^{b_1(M)} \text{vol} H^1(M, \mathbb{Z}) |\text{Hom}(H_1(M, \mathbb{Z})_{\text{tor}}, T)|,
\]

where \(\text{vol} H^1(M, \mathbb{Z})\) denotes the lattice covolume of \(H^1(M, \mathbb{Z}) \subseteq H^1(M, \mathbb{R})\) with respect to the symplectic (resp. Riemannian) structure on \(M\).

Second, for a restricted base manifold \(M\) and a general structure group \(G\).

**Theorem 10.2** ([3]). Let \(M\) be a symplectic (resp. Riemannian) manifold with free abelian fundamental group \(\pi_1(M)\), \(G\) a compact connected semisimple Lie group of dimension \(k\) and rank \(\ell\), \(\langle , \rangle\) an Ad-invariant metric on the Lie algebra \(g\), \(W\) the Weyl group of \(G\) with respect to some maximal torus, \(\{\alpha\} \subseteq H^*\) the root system, and \(\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha\) the half sum of a subsystem of positive roots. Then

\[
\text{vol} \mathcal{M}_G(M) = \left(\frac{\text{vol} G}{\sqrt{2\pi}^{\ell-k}} \prod_{\alpha > 0} \alpha \rho \right)^{b_1(M)} \frac{1}{|W|} \text{vol} H^1(M, \mathbb{Z}),
\]

where \(\text{vol} H^1(M, \mathbb{Z})\) denotes the lattice covolume of \(H^1(M, \mathbb{Z})\) in \(H^1(M, \mathbb{R})\).
In each case, the result is established by working with the representation variety \( \text{Hom}(\pi_1(M), G)/G \), which is diffeomorphic to \( \mathcal{M}_G(M) \). The conditions on \( M \) and \( G \) allow us to exchange \( \text{Hom}(\pi_1(M), G) \) for \( \text{Hom}(H^1(M), G) \), and twisting \( H^1(M) \) by the adjoint representation of \( G \) yields a model for the tangent fibers of \( \mathcal{M}_G(M) \).

Setting \( G = U(1) \) yields the following.

**Corollary 10.2** ([3]). The moduli space of complex line bundles over a manifold \( M \) with flat connection has volume

\[
\text{vol} \, \mathcal{M}_{U(1)}(M) = (2\pi)^{b_1(M)} \text{vol} \, H^1(M, \mathbb{Z}) \left| \text{Ch}(H_1(M, \mathbb{Z})_{\text{Tor}}) \right|,
\]

where \( \text{Ch}(H_1(M, \mathbb{Z})_{\text{Tor}}) \) is the set of characters of \( H_1(M, \mathbb{Z})_{\text{Tor}} \).

### Immersions of the moduli space

The moduli space over a surface \( \mathcal{M}_G(\Sigma) \) has been extensively studied. In [3], I show that under suitable conditions the restriction of connections over \( M \) to a surface \( \Sigma \subset M \) forms a symplectic immersion of \( \mathcal{M}_G(M) \) in \( \mathcal{M}_G(\Sigma) \).

**Theorem 11.2** ([3]). If \( n \geq 2 \), then there is a compact, connected embedded surface \( \Sigma \subset M \) such that \( \Sigma \in H^2(M) \) is the Poincaré dual of \( \eta \). The inclusion \( i : \Sigma \hookrightarrow M \) yields a symplectic immersion \( i^* : \mathcal{M}_G(M) \to \mathcal{M}_G(\Sigma) \). At a connection \( A \) on \( M \), the codimension of the image is equal to

\[
\dim \ker \left( H^2_A(M, \Sigma; \text{ad} \, g) \to H^2_A(M, \text{ad} \, g) \right).
\]

### 3 Eigenvalues of the \( p \)-Laplacian

In [5], S. Seto and I obtain a lower bound for the first eigenvalue of the \( p \)-Laplacian on a Kähler manifold. The \( p \)-Laplacian is a nonlinear differential operator given by

\[
\Delta_p(f) = \text{div}(|\nabla f|^{p-2} \nabla f).
\]

It describes a diffusion process \( \partial_t f = \Delta_p f \) with diffusivity equal to a power of the speed \( |\nabla f| \). When \( p = 2 \), it is the ordinary Laplacian. The eigenvalue condition is given by

\[
\Delta_p(f) = -\mu |f|^{p-2} f.
\]

When \( M \) is closed, the first eigenvalue satisfies the following variational characterization,

\[
\mu_{1,p} = \inf \left\{ \frac{\int_M |\nabla f|^p}{\int_M |f|^p} \mid f \in W^{1,p}(M) \setminus \{0\}, \int_M |f|^{p-2} f = 0 \right\}.
\]

When \( \partial M \neq \emptyset \) and we impose Dirichlet boundary conditions,

\[
\lambda_{1,p} = \inf \left\{ \frac{\int_M |\nabla f|^p}{\int_M |f|^p} \mid f \in W^{1,p}_c(M) \setminus \{0\} \right\}.
\]


In [5], we specialize to Kähler manifolds.
Theorem 1.1 ([5]). Let $(M, J, g)$ be an $n = 2m$ (real) dimensional Kähler manifold, possibly with boundary. Assume that the underlying (real) Ricci curvature satisfies $\text{Ric} \geq Kg$ for some constant $K > 0$. If $\partial M = \emptyset$, then for $p \geq 2$,

$$\frac{p}{2} \mu_{1,p}^2 \geq \frac{p + 2}{p(p - 1)} K = \left( 1 + \frac{2}{p} \right) \frac{K}{p - 1}.$$ 

If $\partial M \neq \emptyset$, we assume the convexity condition that $\frac{p}{2} H + \Pi(Jn, Jn) \geq 0$ and the Dirichlet boundary condition, where $n$ is the unit outward normal vector field on $\partial M$, $H$ is the mean curvature, and $\Pi$ is the second fundamental form. Then for $p \geq 2$,

$$\frac{p}{2} \mu_{1,p}^2 \geq \frac{p + 2}{p(p - 1)} K.$$ 

In the course of the proof we also establish the following $p$-Reilly formula.

Lemma 2.2 ([5]). For $f \in C^2(M)$ and $p \geq 2$,

$$\int_{\partial M} |\nabla f|^p - \left\{ - (\Delta_{\partial M} f + H \nabla_n f) \nabla_n f - \Pi(\nabla_{\partial M} f, \nabla_{\partial M} f) + \langle \nabla (\nabla_n f), \nabla f \rangle_{\partial M} \right\}$$

$$= (p - 2) \int_M |\nabla f|^p - |\nabla^2 f|^2 - \int_M (\Delta f)(\Delta f)$$

$$+ \int_M |\nabla f|^p - 2 |H f|^2 + \text{Ric}(\nabla f, \nabla f) + \langle \text{Hess} f, J^* \text{Hess} f \rangle.$$ 

References


