Magnetic switching of ferromagnetic thin films under thermal perturbation

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In this paper, we study the magnetic switching of submicron-sized ferromagnetic thin films under thermal noise and external field. It is shown that the presence of the noise makes the switching easier with weaker external fields and induces more intermediate metastable states in the switching pathways. Different switching pathways are preferred at different temperatures. A quantitative relation between the temperature and the switching field for different metastable states is given through an adjusted Arrhenius formula near the critical field. Based on this, preferred switching pathways at different temperatures are obtained by comparing the energy barriers along different pathways. © 2005 American Institute of Physics. [DOI: 10.1063/1.1988971]

I. INTRODUCTION

The process of magnetic switching of nanoscale ferromagnetic materials is a very important subject in the study of the magnetic recording process. Applications include technologies of manufacturing computer disks and memory cells, such as magnetoresistive random access memory (MRAM). As the size of the magnetic devices decreases, thermal fluctuations become significant and must be included in realistic analysis and simulations. An attempt to incorporate thermal effects in micromagnetics was done by Brown, Jr. in (Ref. 3) for the special case of single-domain particles which have uniform magnetization. In Refs. 4 and 5, the hysteresis loops of single-domain particles at finite temperature were studied with large deviation theory. For more general cases, experiments and numerical simulations have been extensively carried on to understand the mechanism of the process (see Refs. 6–8 and the references therein) and various observations have been obtained. On the other hand, the magnetic switching under external field is known to be very nonlinear and hysteretic. So what is obviously of interest is the switching of magnetic field under the influences of both thermal perturbation and external field. Problems under investigation are the switching fields and the switching pathways at finite temperature in the hysteresis loops.

The paper is organized as follows. In Sec. II, we will show the thermal effects on the hysteresis loops of ferromagnetic thin films by solving the full stochastic Landau-Lifshitz (SLL) equation. It is observed that under thermal noise, the magnetization switches in the hysteresis loops with weaker external fields than without the noise. The noise may induce more intermediate stages in the switching pathways by driving the process into adjacent metastable states. And at different temperatures, the switching follows different pathways. Next, in Sec. III, we will analyze the overdamped SLL equation for single-domain particles and give a quantitative relation between the strength of the noise and the switching field for different metastable states using an adjusted Arrhenius formula near the critical field. We will also show that the system can be approximated by a reduced discrete Markov chain switching between metastable states. Finally, in Sec. IV, we will apply the same method developed in Sec. III to ferromagnetic thin films to predict the switching field between different metastable states in the hysteresis loops at finite temperature and give the preferred pathways at different temperatures. This is done by using the zero-temperature string method for micromagnetics to find the energy barriers between different metastable states.

II. MICROMAGNETICS UNDER THERMAL NOISE

A. Dynamics and numerical schemes

Based on the Landau-Lifshitz theory, the dynamics of the magnetization distribution in a ferromagnetic material is described by the following Landau-Lifshitz equation:

$$\dot{M} = -\gamma M \times H - \frac{\gamma \alpha}{M_s} M \times (M \times H),$$

(1)

where $|M|=M_s$(const) is the saturation magnetization far from the Curie temperature. $\alpha$ is a dimensionless damping coefficient. $\gamma=ge/(2m_e)$ is the gyromagnetic ratio where $e$ and $m_e$ are the positive charge and mass of the electron. $H$ is the local field computed as the following unconstrained first variation:

$$H = -\frac{\delta F}{\delta M},$$

(2)

where $F$ is the Landau-Lifshitz energy functional.
\[ F[M] = \frac{1}{2} \int_{\Omega} \left\{ \frac{C_{ex}}{M_z^2} \nabla M \nabla M^2 + \Phi \left( \frac{M}{M_z} \right) \right\} \, dx. \]

\[ -2\mu_0 H_e M + \mu_0 \nabla U \nabla U \right\} \, dx. \] (3)

\( \Omega \) is the volume occupied by the material. \( C_{ex} \) is the exchange constant, and \( \Phi(M/M_z) = K_a(M_z^2 + M_i^2)/M_z^2 \) is the anisotropy. \(-2\mu_0 H_e M \) is the energy due to the external field \( H_e \), where \( \mu_0 \) is the permeability of vacuum. The last term in the energy is due to the field induced by the magnetization inside the material such that

\[ \Delta U = \begin{cases} \nabla M, & x \in \Omega \\ 0, & x \in \Omega^c \end{cases}, \] (4)

together with the jump condition at the boundary \( [U] = 0 \), \( [\nabla U \cdot \nu] = -M \cdot \nu \).

To introduce the thermal effects, we replace \( H \) in (1) with \( H + \sqrt{\kappa} \dot{W} \) where \( w \) is a space-time white noise and \( \dot{W} \) denotes the Stratonovich integral in time. By the fluctuation-dissipation theorem, the strength of the noise should satisfy

\[ \kappa = \frac{2\alpha K_B T}{(1 + \alpha^2) \gamma M_z^2}, \] (5)

where \( K_B \) is the Boltzmann constant and \( T \) is the temperature. Then Eq. (1) becomes the SLL equation,

\[ M = -\gamma \times M(H + \sqrt{\kappa} \dot{W}) - \frac{\gamma \alpha}{M_z} M \times M \times (H + \sqrt{\kappa} \dot{W}). \] (6)

It is shown in Appendix A that the strength of the noise we add as in (5) is consistent with the case when the thin-film sample is reduced to single-domain particles, for which both the magnetization and stray field are uniform: therefore, the exchange energy vanishes. This happens when the size of the material sample is very small. We apply a quasistatic external field such that

\[ H_e = (-1 + \gamma|t|/|\Delta|)H_{\text{max}}, \quad t \in \left[ 0, \frac{2}{\gamma} \right], \] (7)

where \( \Delta \) is the time interval during which each external field is applied. We use \( \lfloor x \rfloor \) to denote the biggest integer no larger than \( x \) so the external field applied as above is changed quasistatically with a constant value on each subinterval \( [k \Delta, (k+1)\Delta) \). \( \gamma > 0 \) measures the speed with which the external field is changed.

We solve the Landau-Lifshitz equation [Eq. (1)] with an Euler-projection scheme. The Euler method is adopted for the time discretization and the magnetic field is renormalized to the sphere of constant magnetization at each time step:

\[ M^* = M_{\text{is}} + \Delta t \left( -\gamma M_{\text{is}} \times H_{\text{is}} - \frac{\gamma \alpha}{M_z} M_{\text{is}} \times M_{\text{is}} \times H_{\text{is}} \right), \]

\[ M_{\text{is+1}} = M_{\text{is}^*}/|M^*|. \] (8)

For the space discretization, we divide the computational domain into cells. In each cell we approximate the magnetization by a vector with constant magnitude, but free to rotate in any direction. We approximate the stray field by its average value in each cell. This stray field is computed using fast Fourier transform (FFT). The details of this computation can be found in (Ref. 10). We also solve the stochastic Landau-Lifshitz equation [Eq. (6)] with the Euler-projection scheme:

\[ M^* = M_{\text{is}} + \Delta t \left( -\gamma M_{\text{is}} \times \tilde{H}_{\text{is}} - \frac{\gamma \alpha}{M_z} M_{\text{is}} \times M_{\text{is}} \times \tilde{H}_{\text{is}} \right), \]

\[ M_{\text{is+1}} = M_{\text{is}^*}/|M^*|. \] (9)

where

\[ \tilde{H}_{\text{is}} = H_{\text{is}} + \sqrt{\kappa} \nu \Delta w_{\text{is}}, \] (10)

with \( \nu \) being the volume of the computational cell. The time discretization for the thermal noise is performed in the following Stratonovich sense:

\[ \Delta w_{\text{is}} = \frac{w_{\text{is+1}} - w_{\text{is}}}{\sqrt{2}}, \] (11)

where \{\( w_{\text{is}} \)\}'s are independent and identically distributed standard random walks on the real line.

We choose the sample to be a \( 200 \times 200 \times 10 \) nm\(^3\) square permalloy film and the computational grid to be \( 64 \times 64 \). The time step is chosen to be \( 10^{-13} \) s for both the Landau-Lifshitz equation [Eq. (1)] and the stochastic Landau-Lifshitz equation [Eq. (6)]. We use the same physical parameters as in Ref. 7 such that

\[ \alpha = 1, \]

\[ \mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2, \quad \gamma = 1.76 \times 10^{11} \text{ T}^{-1} \text{s}^{-1}, \]

\[ M_z = 9.0 \times 10^5 \text{ A/m}, \quad K_a = 1.0 \times 10^2 \text{ J/m}^3, \]

\[ C_{ex} = 1.3 \times 10^{-11} \text{ J/m}, \quad K_B = 1.380 \times 65 \times 10^{-23} \text{ J/K}. \] (12)

The external field is changed along the direction (cos \( \theta_0, \sin \theta_0, 0 \)), \( (0 < \theta_0 \leq 1) \) from \( -H_{\text{max}} = -300 \) Oe to \( H_{\text{max}} = 300 \) Oe. For the loop without noise, we initialize the magnetization uniformly with \((-1, 0, 0)\) in the loop. Each exter-
nal field is run until the process reaches a stable state. This is guaranteed by small thresholds for the magnitudes of the gradient field $\bar{\delta} F / \bar{\delta} \mathbf{M}$ and the relative changes of $F$ and $M$. For the loop with noise, we initialize the magnetization with the stable state for the initial external field obtained in the loop without the noise. For each external field, the stochastic Landau-Lifshitz dynamics is run for 5 ns.

B. Hysteresis loops at zero and finite temperature

For ferromagnetic thin films, there are two stable configurations at zero temperature, commonly known as $S$ state and $C$ state. For $S$ states, the magnetization in two end domains is parallel to each other and the magnetization forms an S-shape configuration. For $C$ states, the magnetization in two end domains is opposite to each other and the magnetization forms a C-shape configuration. In Fig. 1, we show the hysteresis loops without and with the thermal noise at temperature $T=300$ K. The switching in the zero-temperature loop has two phases. The first one is from $S$ stage $S_1$ to $S$ stage $S_2$ and the second one is from $S$ stage $S_2$ to another $S$ stage $S_3$. At finite temperature, there are two thermal effects on the loop. First, the switching occurs at a lower field in the $S_1 \rightarrow S_2$ switching. Second, with the noise, the magnetization switches along the $S_2 \rightarrow C_1 \rightarrow S_3$ pathway, with one more intermediate $C$ state stage $C_1$. We apply decreasing/increasing external fields on stage $C_1$ without noise and get the stable states at zero temperature, which is shown in Fig. 2.

In Fig. 2, we also show the loops for different realizations of the thermal noise at the same temperature $T=300$ K. It can be seen that the thermal effect that the magnetization switches with a weaker field does not change with respect to realizations. But instead of switching to $C_1$, for some realizations with the same probability, the magnetization switches to another different $C$ state stage $C_2$. For each external field, states in $C_1$ and $C_2$ stages have the same mean magnetization. Figure 3 gives the $S$ state in stage $S_1$ and the $C$ states in stages $C_1$ and $C_2$. It can be seen that there is a symmetry between $C_1$ and $C_2$ with respect to $S_2$. The magnetization in the top domains of $S_2$ and $C_1$ are opposite and the magnetization in the bottom domains of $S_2$ and $C_1$ are parallel. Meanwhile, the magnetization in the bottom domains of $S_2$ and $C_2$ are opposite and the magnetization in the top domains of $S_2$ and $C_2$ are parallel. In other words, from $S_2$, the magnetization switches to $C_1$ if the top domain of $S_2$ changes the direction and switches to $C_2$ if the bottom domain of $S_2$ changes the direction. For very few realizations, the intermediate $C$ states are not observed. The same observation holds for temperatures up to $T=900$ K. We give in Fig. 4 the loops from 0 to 900 K. It can be seen that the higher the temperature, the weaker the external field needed for the $S_1 \rightarrow S_2$ switching is.

Now we raise the temperature up to $T=1200$ K to see the thermal effect on the hysteresis loops at higher temperatures. In Fig. 5, we show some realizations of the loops at $T=1200$ K in which the switching follows the $S_2 \rightarrow C_1/C_2 \rightarrow S_3$ pathways. Consistent with the above observations, the $S_1 \rightarrow S_2$ switching happens at much lower fields. The only difference from the lower temperatures is that after the magnetization settles at the $S_3$ stage, it may switch to another $C$ state which has a mean magnetization close to that of the $S_3$ states. Figure 6 shows a different switching pattern for other realizations at $T=1200$ K. The magnetization first switches

![Fig. 2. Hysteresis loops of thin film for different realizations of the noise at $T=300$ K.](image)

![Fig. 4. Hysteresis loops of thin film at different temperatures.](image)

![Fig. 3. S state in stage $S_1$ and $C$ states in stage $C_1$ and $C_2$.](image)
at a negative external field to a $C$ state. There are two different configurations for this $C$ state, which we denote by $C_A$ and $C_B$. Then at a positive external field, the magnetic field switches to a vortex state $V_A$ or $V_B$, whose configuration depends on the previous $C$ state. After the vortex state, the magnetization will switch to another $C$ state stage $C'_A$ or $C'_B$ depending on the path it has followed. Again, after the $C'_A/C'_B$ stages, the magnetization may switch to another $C$ or $S$ state with a close mean magnetization. In short, for these realizations, the switching follows the $S$ state with a close mean magnetization. In short, for these realizations, the switching follows the $S$ state with a close mean magnetization. In short, for these realizations, the switching follows the $S$ state with a close mean magnetization. In short, for these realizations, the switching follows the $S$ state with a close mean magnetization. In short, for these realizations, the switching follows the $S$ state with a close mean magnetization.

In conclusion, the above results show that the thermal noise makes the magnetic switching easier with weaker external fields and induces different switching pathways. The higher the temperature, the more different pathways the switching may follow. A quantitative study will be given in subsequent sections.

III. MAGNETIC SWITCHING OF SINGLE-DOMAIN PARTICLES

In this section, we want to give quantitatively the relation between the strength of thermal noise and the switching field for different metastable states in the hysteresis loops of single-domain particles in which the magnetization is uniform. This relation is given through Eq. (22). If $\alpha \gg 1$, the first term (the gyromagnetic term) on the right-hand side of the Landau-Lifshitz equation [Eq. (1)] is dominated by the second term (the damping term) and hence can be dropped.

Transforming into angular variables and introducing the thermal noise, we get the following random perturbed gradient system describing the overdamped Landau-Lifshitz dynamics for single-domain particles under thermal noise and changing external field:

$$\dot{\phi} = -\frac{1}{\delta} \nabla \phi V(\phi, t) + \sqrt{\frac{2e}{\delta}} \omega,$$

where

$$V(\phi, t) = -\frac{1}{2} \cos 2(\phi - \theta_0) - h_0(-1 + [t/\Delta]|\Delta|) \cos \phi.$$

Here $\phi$ represents the angle between the magnetization and the external field. $\delta$ is a constant obtained from variable transformation and depends on the physical parameters of the system. $\varepsilon$ is the strength of the noise determined by fluctuation-dissipation theorem and $\omega$ is a standard Brownian motion. The energy function $V(\phi, t)$ is chosen to be the Stoner-Wohlfarth potential for single-domain ferromagnetic materials. It is a special case of the Landau-Lifshitz potential when the magnetization is uniform across the sample. We are assuming no crystalline anisotropy here for simplicity and the fact that it is usually smaller than the shape anisotropy in permalloy. The first term of $V(\phi, t)$ is the anisotropy and the second is the energy due to the following time-dependent external field:

$$h = h_0(-1 + [t/\Delta]|\Delta|), \quad t \in [0, 2],$$

which is changed from $-h_0$ to $h_0$. $\Delta$ represents the observation time for each external field. $[x]$ is defined as before to be the biggest integer no larger than $x$. Notice that we can also write $V = V_h(\phi)$. $\theta_0$ gives the preferred direction of magneti-

FIG. 7. $C$ and vortex states in $C_A \rightarrow V_A \rightarrow C'_A$ pathway with increasing $H_x$.

FIG. 5. Hysteresis loops at $T=1200$ K following the $S_1 \rightarrow C_A/C_2 \rightarrow S_3$ pathways.

FIG. 6. Hysteresis loops at $T=1200$ K following the vortex pathways.
steady states of one is the relaxation time scale for the process to reach the local minima of $V_h(\phi)$. gives the hysteresis loops generated by the separation of the driven by the noise. This time scale depends on the strength of the noise. The fourth time scale is the switching perturbation on the hysteresis loops. For numerical solutions of

$$h_c = 0.590,$$

which is the switching field in the hysteresis loop without noise, we have the following:

1. For $h \leq -h_c$ and $h > h_c$, $V_h(\phi)$ has one minimum and one maximum, and
2. For $-h_c < h < h_c$, $V_h(\phi)$ has two local minima and two local maxima.

The energy landscapes of $V$ for different values of $h$ are shown in Fig. 10. Due to symmetry, we only analyze the switching near the critical field $h_c$ when the external field is applied from $-1$ to $1$. For $-h_c < h < h_c$, we denote by $\psi_1(h)$ the local minimum of $V_h(\phi)$ shown in Fig. 9 as the bottom loop without noise when $h$ is changed from $-1$ to $1$ and denote by $\psi_2(h)$ the other local minimum. We also denote by $\theta_1(h)$ and $\theta_2(h)$ the local maxima. Let $\theta(h) \in \{\theta_1(h), \theta_2(h)\}$ be the critical point with a lower-energy barrier such that $V_h[\theta(h)] = \min\{V_h[\theta_1(h)], V_h[\theta_2(h)]\}$. We define the energy barrier $\Delta V(h)$ from $\psi_1$ to $\psi_2$ to be

$$\Delta V(h) = V_h[\theta(h)] - V_h[\psi_1(h)].$$

Numerical results given in Fig. 11 show that $\Delta V(h)$ is monotonically decreasing with respect to $h$ on the interval $[-h_c, h_c]$.

Notice that on each time subinterval $[i\Delta, (i+1)\Delta)$, dynamics (13) is homogeneous, i.e., the coefficients are time

FIG. 9. Hysteresis loops of single-domain particle with and without the noise.

FIG. 10. Energy landscapes of the single-domain particles for different external fields.
independent. It is shown in Appendix B that for each external field near \( h_c \), the mean exit time \( \tau \) for the process to overcome the energy barrier \( \Delta V \) to switch from \( \psi_1 \) to \( \psi_2 \) is given by the following adjusted Arrhenius formula:

\[
\tau(h) = \frac{\nu_0 \delta}{\epsilon^{1/3}} \exp \left( \frac{\Delta V(h)}{\epsilon} \right),
\]

where \( \nu_0 = \int_0^\infty e^{-x^2/6} \, dx \sqrt{3} [\lim_{h \to -h_c} \psi_1(h)]^{2/3} \). Direct numerical computation shows that \( \nu_0 = 2.127 \). Due to the exponential dependence of the exit time scale on \( \Delta V \) and the monotonicity of \( \Delta V \), the exit time scale decreases when \( h \) approaches \( h_c \). The switching should be most probable when the exit time scale is comparable to the observation time scale such that

\[
\tau(h^*) = \Delta.
\]

The monotonicity of \( \Delta V(h) \) implies an inverse function \((\Delta V)^{-1}\) for \( \Delta V \) and a unique solution for Eq. (21). Then we have the following equation for the switching field in the hysteresis loops under random perturbation:

\[
h^* = (\Delta V)^{-1} \left[ \epsilon \ln \left( \frac{\epsilon^{1/3} \Delta}{\nu_0 \delta} \right) \right].
\]

We solve the Fokker-Planck equation to get the loops for different strengths of the noise and pick up the switching field in the loop to be the first field where the magnetization is fully switched. Table I gives this result and the switching field predicted by (22). The relative error is < 0.03. The hysteresis loop and the predicted switching field when \( \epsilon = 0.01 \) are shown in Fig. 12.

By the fact that the dynamics is homogeneous on each subinterval, we know that for each external field, the \( n \text{th} \) (\( n \in \mathbb{N} \)) moment \( \tau_n(\phi) \) of the exit time satisfies the equation

\[
\frac{1}{n} \frac{\partial V}{\partial \phi} \frac{\partial}{\partial \phi} \tau^2(\phi) + \frac{\epsilon^2}{\delta} \frac{\partial V}{\partial \phi} \frac{\partial^2}{\partial \phi^2} \tau^2(\phi) = -n \tau^{n-1}(\phi).
\]

Boundary layer analysis as in Ref. 13 can show that except on a boundary layer of thickness of \( n \epsilon \),

\[
\tau^2(\phi) = (nK)\tau^{n-1}(\phi) = \text{const.}
\]

This means that away from the boundary, the exit time can be approximated by an exponentially distributed random variable with parameter \( \tau(\phi) = K \) in the sense that the moments are asymptotically close. Similar results are also given in Ref. 14 by a more subtle analysis. The above analysis implies that we can approximate dynamics (13) by a discrete Markov chain in which the process switches between \( \psi_1(h) \) and \( \psi_2(h) \) with exponentially distributed transition probabilities, which provides a method to dramatically reduce the computation by simulating the Markov chain instead of solving the Fokker-Planck equation. Figure 13 gives the expectations of the magnetization with respect to this Markov chain using the mean exit time \( \tau \) given by (20) when \( \epsilon = 0.01 \). It can be seen that it is almost identical with the result by solving the Fokker-Planck equation.

### IV. SWITCHING OF FERROMAGNETIC THIN FILMS

From Sec. II, we see that the thermal noise has two effects on the magnetic switching. The first is to make the switching easier with weaker external fields. The second is to induce different switching pathways at different temperatures. The questions that need to be answered are what the switching field under noise is and which pathways are preferred at different temperatures. In this section, we will apply the adjusted Arrhenius formula [Eq. (20)] to these problems.

#### A. Energy landscapes under different external fields

First we want to study the energy landscapes of thin films under different external fields and the implications in magnetic switching. For a given energy function or functional \( V(x) \), the minimum energy path (MEP) \( \phi \) between different metastable states \( A \) and \( B \) is defined to be the curve connecting \( A \) and \( B \) and satisfying the following equation:\( \frac{1}{n} \frac{\partial V}{\partial \phi} \frac{\partial}{\partial \phi} \tau^2(\phi) + \frac{\epsilon^2}{\delta} \frac{\partial V}{\partial \phi} \frac{\partial^2}{\partial \phi^2} \tau^2(\phi) = -n \tau^{n-1}(\phi) \).
\[ \nabla^2 V(\varphi) = 0, \]  

where \( \nabla \) denotes the projection onto the hyperplane perpendicular to \( \varphi \). It is known\(^{16}\) that, for Landau-Lifshitz potential and dynamics, the MEP shares the same critical points and hence the same energy barriers with the transition pathways between different metastable states at finite temperature. The zero-temperature string method for micromagnetics\(^{16}\) solves the MEP by evolving the following gradient flow in the path space:

\[ \dot{\phi}_t = -\nabla^2 V(\phi) + r\dot{r}, \]  

where \( \phi = \phi_{[0,1]} \) is a curve connecting \( A \) and \( B \). \( \dot{r} = \phi_T / |\phi_T| \) is the unit tangent along \( \phi \), \( r \) is a Lagrangian multiplier which keeps a certain parametrization of the evolving curve. In the following, we are going to choose \( r \) such that

\[ |\phi_T|/|\phi_T| = \sin(\alpha \pi/2), \quad \alpha \in [0,1], \]  

where \( |\phi_T| \) is the arclength of the evolving curve. \( \eta > 1 \) is a fixed number. This parametrization allows to focus on the nucleation of the switchings. The string method can be easily parallelized by dividing the evolving curve into a certain number of subcurves, all evolving with (26). This can be implemented by an message passing interface (MPI) structure.

In Fig. 14, we give the energy barriers between \( S_1 \) and \( S_2 \) stages for different external fields.

### B. Switching field at finite temperature

Now we want to study the switching field in the hysteresis loops of thin films with the adjusted Arrhenius formula. We do the following substitution for (20) in the context of the stochastic Landau-Lifshitz dynamics:

\[ \delta \to \frac{1}{\alpha \gamma M_S}, \quad \epsilon \to k_BT. \]  

Then we have the adjusted Arrhenius formula for ferromagnetic thin film which gives the mean exit time from the neighborhood of one metastable state near the critical field:

\[ \tau = \frac{\nu_0}{\alpha \gamma M_S (k_BT)^{1/3}} \exp[\Delta F/(k_BT)], \]  

where \( \Delta F \) is the energy barrier between different metastable states. \( \nu_0 \) is a prefactor depending on the third-order curvature of the energy landscape at the critical field. In the hysteresis loops with thermal noise, the switching should happen when the exit time scale is equal to the observation time scale, i.e.,

\[ \tau = \Delta. \]  

Due to the difficulty arising from the infinite dimensional nature of the Landau-Lifshitz dynamics, we still do not have efficient tools to estimate \( \nu_0 \). Since the dependence of \( \tau \) on the energy barrier \( \Delta F \) is exponential and much more significant than the dependence of \( \tau \) on \( \nu_0 \), we assume \( \nu_0 = 1 \). By the monotonicity of \( \Delta F \), we have the following equation for the switching field at temperature \( T \):

\[ H^*_s = (\Delta F)^{-1}(k_BT \ln[\Delta \alpha \gamma M_s (k_BT)^{1/3}]). \]  

### TABLE II. Simulated and predicted switching fields of thin films.

<table>
<thead>
<tr>
<th>Temperature (K)</th>
<th>100</th>
<th>200</th>
<th>300</th>
<th>400</th>
<th>500</th>
<th>600</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean switching field (Oe)</td>
<td>132.7</td>
<td>125.2</td>
<td>124.4</td>
<td>120.7</td>
<td>103.4</td>
<td>96.7</td>
</tr>
<tr>
<td>Predicted field (Oe)</td>
<td>127.1</td>
<td>121.2</td>
<td>116.3</td>
<td>111.7</td>
<td>107.5</td>
<td>103.5</td>
</tr>
</tbody>
</table>
Now we focus on the $S_1 \rightarrow S_2$ switching. We use formula (31) to predict the switching field for temperatures from 100 to 600 K. At the same time, we compute eight realizations of the loops for each temperature and take the mean switching field. Table II and Fig. 15 give the mean switching fields and the predicted fields. The relative error is <0.08.

C. Preferred pathways at different temperatures

Now we want to study the preference of different pathways at different temperatures. We first focus on the lower temperatures. From the results in Sec. II, we know that at zero temperature, the switching follows the $S_2 \rightarrow S_3$ pathway while at finite temperature, the switching follows the $S_2 \rightarrow C_1/C_2 \rightarrow S_3$ pathways. Figure 16 gives the energy barriers of the $S_2 \rightarrow S_3$ and $S_2 \rightarrow C_1$ switchings for different external fields. Due to symmetry, the energy barriers for the $S_2 \rightarrow C_2$ switching will be the same as those for the $S_2 \rightarrow C_1$ switching. It can be seen that the energy barriers for the $S_2 \rightarrow C_1/C_2 \rightarrow S_3$ pathways are much smaller than the barriers of the $S_2 \rightarrow S_3$ switching. This means that the $S_2 \rightarrow C_1/C_2 \rightarrow S_3$ pathways are more preferred than the $S_2 \rightarrow S_3$ pathway.

Now we want to study the preference of different pathways at different temperatures. We first focus on the lower temperatures. From the results in Sec. II, we know that at zero temperature, the switching follows the $S_2 \rightarrow S_3$ pathway while at finite temperature, the switching follows the $S_2 \rightarrow C_1/C_2 \rightarrow S_3$ pathways. Figure 16 gives the energy barriers of the $S_2 \rightarrow S_3$ and $S_2 \rightarrow C_1$ switchings for different external fields. Due to symmetry, the energy barriers for the $S_2 \rightarrow C_2$ switching will be the same as those for the $S_2 \rightarrow C_1$ switching. It can be seen that the energy barriers for the $S_2 \rightarrow C_1/C_2 \rightarrow S_3$ pathways are much smaller than the barriers of the $S_2 \rightarrow S_3$ switching. This means that the $S_2 \rightarrow C_1/C_2 \rightarrow S_3$ pathways are more preferred than the $S_2 \rightarrow S_3$ pathway.

Now we move to the higher temperature of 1200 K. Figure 18 gives the energy barriers for the $S_1 \rightarrow C_A$ pathways for different external fields. Again for the reason of the symmetry between $C_A$ and $C_B$ with respect to $S_1$, the energy barriers for the $S_1 \rightarrow C_B$ is the same. It can be seen that the energy barrier increases when external field is increased from negative to positive. Since this switching is away from the critical field, we can use the original Arrhenius formula without the adjustment in the prefactor:

$$
\tau = \frac{\nu}{\alpha \gamma M_s} \exp[\Delta F/(k_B T)].
$$

Assuming $\nu_0=1$ again in switching condition (30) gives

$$
H^*_e = (\Delta F)^{-1}[k_B T \ln(\Delta \alpha \gamma M_s)].
$$

The predicted switching field for $S_1 \rightarrow C_A/C_B$ at $T=1200$ K using formula (33) is $H^*_{e,1200}=-129.1197$ Oe. At $T=600$ K, the predicted field is $H^*_{e,600}=-348.9502$ Oe. Notice that $H^*_{e,600}<H_{\text{max}}<H^*_{e,1200}$ where $H_{\text{max}}$ is the maximum exter-

![FIG. 15. Simulated and predicted switching field between $S_1$ and $S_2$ stages.](image)

![FIG. 16. Energy barriers of $S_2 \rightarrow S_3$ and $S_2 \rightarrow C_1$ switchings for different external fields.](image)

![FIG. 17. Energy along the MEPs of $S_2 \rightarrow S_3$ and $S_2 \rightarrow C_1 \rightarrow S_3$ switchings when the external field is 90 Oe.](image)

![FIG. 18. Energy barriers for the $S_1 \rightarrow C_A$ switching under different external fields.](image)
nal field applied on the sample. This means that, under the external field we are applying, the $S_1 \rightarrow C_A/C_B$ switching is impossible at $T=600$ K and is more preferred at $T=1200$ K. In other words, when the external field is negatively large and the temperature is high, the energy barriers separating $S_1$ and $C_A/C_B$ are so small that they are ignored by the fluctuation. This also by symmetry explains the observation that the process may switch to other $S$ and $C$ states after $C'_A/C'_B$ when the external field gets positively large and the fluctuation becomes important again. In Fig. 19, we give the energy along the minimum energy path along the $C_A \rightarrow V_A \rightarrow C'_A$ switching when the external field is 180 Oe. The increasing energy barrier also implies that the $S_1 \rightarrow C_A/C_B$ switching becomes difficult when the external field is increased from negative to positive. This is why for some of the realizations, when the $S_1 \rightarrow C_A/C_B$ switching does not happen at large negative external fields, the switching will follow the $S_1 \rightarrow S_2$ pathway later when the external field gets positive.

V. CONCLUSIONS

So far, we have studied the effects of incorporating thermal noise into the full Landau-Lifshitz dynamics for ferromagnetic thin films and the relation between the thermal noise and switching field in the hysteresis loops. The stochastic Landau-Lifshitz equation with Stratonovich noise is solved with an Euler-projection method. The energy barriers between different metastable states under different external fields are obtained with the string method. Hence the switching field can be predicted for different temperatures by the adjusted Arrhenius formula and preferred pathways at different temperatures can be given. Future work involves the asymptotics for the prefactor of the Arrhenius formula in infinite dimensions.

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APPENDIX A: THE STRENGTH OF THE THERMAL NOISE

Here we want to give an argument for the strength of the thermal noise we added in the Landau-Lifshitz equation as given by (5). In Ref. 3, the following Gilbert equation was used to describe the dynamics of the magnetization for single-domain particles:

$$M_t = \gamma_0 M \times \left( -\frac{1}{\nu} \frac{d}{dM} - \eta M_t \right) = \gamma_0 \delta(M - \eta M_t),$$

(A1)

where $F$ is the Landau-Lifshitz potential and $\nu$ is the volume of the particle. Comparing the coefficients with (1) gives

$$\gamma_0 = -\gamma(\alpha^2 + 1), \quad \eta = \frac{\alpha}{\gamma M_0 (\alpha^2 + 1)}.$$  

(A2)

To incorporate the thermal noise, in Ref. 3, the field $H$ in (A1) was replaced by $H + \tilde{h}$, where $\tilde{h}$ is a Stratonovich noise white in time. It is shown$^3$ that for the above equation to have an equilibrium distribution $e^{-\Gamma(M) / (k T)}$, the strength of the noise $\tilde{h}$ should satisfy the following:

$$\langle \tilde{h} \rangle = 0, \quad \langle \tilde{h}_{\alpha} \tilde{h}_{\beta} \rangle = \frac{2 k_B T}{\nu} \delta_{\alpha \beta} = \frac{2 \alpha k_B T}{\gamma M_0 (\alpha^2 + 1)} \delta_{\alpha \beta}, \quad D \delta_{\alpha \beta}.$$

(A3)

If we add the noise in (6) such that $\kappa = 2 \alpha k_B T / \gamma M_0 (\alpha^2 + 1)$. Then after space discretization, the noise on each computational cell $k$ satisfies

$$\langle w_\alpha \rangle = 0, \quad \langle w_{\alpha} w_{\beta} \rangle = D \delta_{\alpha \beta} \delta(\tau),$$

(A4)

which is consistent with (A3) and fluctuation-dissipation theorem.

APPENDIX B: ADJUSTED ARRHENIUS FORMULA NEAR THE CRITICAL FIELD

In this Appendix, we want to give the adjusted Arrhenius formula for the single-domain particles near the critical field $h_c$. Notice that by the definition $V_0(\theta(h)) = \min[V_0(\theta_1(h)), V_0(\theta_2(h))]$. With a much higher probability, the process switches by overcoming $\theta(h)$ and $\Delta V(h)$ instead of overcoming the other critical point $\{\theta_1, \theta_2\}$, which has a higher energy barrier. Hence we can impose a reflecting boundary condition at $\{\theta_1, \theta_2\}$. Solving directly (Eq. (23)) for $n = 1$, $^{12}$ we have

$$\tau = \delta \int_{\psi_1}^{\psi_2} \int_{\theta_1}^{\theta_2} d\alpha \exp[V(\alpha)/\epsilon] \int_0^\theta d\beta \exp[-V(\beta)/\epsilon].$$

(B1)

It can be seen that when $h \rightarrow h_c^+, \theta(h) \rightarrow \theta^+$ and $\psi(h) \rightarrow \theta^+$ for some $\theta^+$. Notice that by the stability condition,
FIG. 20. Energy landscape for single-domain particle near the critical field.

\[ V_h^i(\theta(h)) = V_h^i(\psi_1(h)) = 0, \quad V_h^c(\theta(h)) = 0 \leq V_h^c(\psi_1(h)). \]  

(B2)

By continuity, we have for \( i = 1, 2, \)

\[ V_h^{(i)}(\theta') = \lim_{{h \to h^-_c}} V_h^{(i)}(\theta(h)) = \lim_{{h \to h^-_c}} V_h^{(i)}(\psi_1(h)) = 0, \]  

(B3)

while direct computation shows that \( V^{(3)}(\theta') < 0. \) We make the following third-order Taylor expansion for \( \psi_1 < \theta < \psi_2: \)

\[ V_h(\phi) = V_h(\theta) + \frac{1}{2} V_h^{(2)}(\theta)(\phi - \theta)^2 + \frac{1}{6} V_h^{(3)}(\theta)(\phi - \theta)^3 \]

\[ = V_h\left(\psi_1\right) + \frac{1}{6} V_h^{(3)}\left(\theta'\right)(\phi - \theta')^3, \]  

(B4)

and for \( \{\psi_1, \psi_2\}/\phi < \theta < \phi, \) we have

\[ V_h(\phi) = V_h\left(\psi_1\right) + \frac{1}{6} V_h^{(3)}\left(\theta'\right)(\phi - \theta')^3. \]  

(B5)

From Fig. 20, we can see that we only need to integrate on half of the real line. Hence we have

\[
\tau = \frac{\delta}{\epsilon} \exp \left[ \frac{V_h(\theta) - V_h(\psi_1)}{\epsilon} \right] 
\times \int_{\phi'}^\infty \exp \left[ -\frac{1}{6} V_h^{(3)}(\theta')(\phi - \phi')^3/\epsilon \right] d\phi 
\times \int_{-\infty}^\phi \exp \left[ -\frac{1}{6} V_h^{(3)}(\theta')(\phi - \theta')^3/\epsilon \right] d\phi 
\times \left( \frac{1}{\epsilon} \right)^{1/3} \left[ V_h^{(3)}(\theta') \right]^{2/3} \exp \left( \frac{\Delta V}{\epsilon} \right). \quad (B6)
\]