Exercise3 (p.101). Prove that if $H \lhd G$ has prime index p then for all $K \leq G$ either

(i) $K \leq H$ or

(ii) $G = HK$ and $[K : K \cap H] = p$.

Solution. If $K \leq H$ then we are done. So assume that $K \nleq H$ and thus there exists $k \in K \backslash H$. Since H is normal in G the quotient G/H is a group. In particular, it has order p by hypothesis, and so it must be cyclic of order p and is generated by any non-identity element (see Corollary 10 on p.90 in the textbook). Since $k \notin H$ implies $kH \neq H$, we have

$$G/H = \langle kH \rangle \qquad (*).$$

Now, to show $G = HK$, let $g \in G$ and consider $gH \in G/H$. By $(*)$, we know that

$$gH = (kH)^r = k^r H \quad \text{for some } r \in \mathbb{Z}.$$

It follows that $k^{-r} g \in H$, say $k^{-r} g = h$ with $h \in H$. Then

$$g = k^r h \in KH = HK$$

(the last equality follows because H is normal in G). This shows that $G \subset HK$. This other inclusion is clear since $H, K \subset G$. Therefore, indeed $G = HK$.

To show that $[K : K \cap H] = p$, we use the second isomorphism theorem. It tells us that

$$\frac{KH}{H} \simeq \frac{K}{K \cap H} \text{ and so in particular } [KH : H] = [K : K \cap H].$$

Since $KH = G$, we have $[K : K \cap H] = [G : H] = p$, as desired.