MATH 111A HOMEWORK 10 SOLUTIONS

---

13) Find all finite groups which have exactly two conjugacy classes.

Proof. Let G be a finite group with exactly two conjugacy classes. Since $\{1\}$ is conjugacy class, and the conjugacy classes partition G, the two conjugacy classes must be

$$\{1\} \quad \text{and} \quad G - \{1\}.$$

Let $g \in G - \{1\}$. Then, by the orbit-stabilizer formula, we have

$$\begin{aligned} |G - \{1\}| &= |\text{conjugacy class containing } g| \\ &= |\text{orbit of } g \text{ under the conjugation action}| \\ &= [G : \mathrm{stab}_G(g)]. \end{aligned}$$

Since $[G : \mathrm{stab}_G(g)]$ divides $|G|$ by Lagrange's theorem, we deduce that

$$|G - \{1\}| \text{ divides } |G| \implies |G| - 1 \text{ divides } |G|.$$

This is only possible if $|G| = 2$. Thus, we have $G \simeq \mathbb{Z}/2\mathbb{Z}$.

---

15) Prove that a group of order 351 has a normal Sylow p-subgroup for some prime p dividing its order.

Proof. Let G be a group of order $351 = 3^3 \cdot 13$. Let n_{13} be the number of Sylow 13-subgroups in G. Then, Sylow's Theorem Part III tells us that

$$n_{13} \text{ divides } 3^3 \quad \text{and} \quad n_{13} \equiv 1 \pmod{13}.$$

Hence, either $n_{13} = 1$ or $n_{13} = 27$. If $n_{13} = 1$ then the unique Sylow 13-subgroup is normal and we are done. So suppose that $n_{13} = 27$ and let $P_1, ..., P_{27}$ be the distinct Sylow 13-subgroups. Observe that

(i) $P_i \cap P_j = \{1\}$ for all $i \neq j$: This is because

$$P_i \cap P_j \leq P_i \implies |P_i \cap P_j| \text{ divides } |P_i|.$$

But $|P_i| = 13$ is prime so $|P_i \cap P_j| = 1$ or 13. It cannot be 13 for it would imply that

$$P_i \cap P_j = P_i \implies P_i \subset P_j \implies P_i = P_j$$

since P_i and P_j have the same number of elements.

(ii) All non-identity elements in P_i have order 13: This is because $P_i \simeq \mathbb{Z}/13\mathbb{Z}$ (from a previous homework) is cyclic of prime order (see also the Corollary 10 on p.90).

From these two facts we see that there are $27 \times 12 = 324$ elements of order 13. The remaining

$$351 - 324 = 27$$

elements must form a unique Sylow 3-subgroup (unique because elements in a Sylow 3-subgroup cannot have order 13 by Lagrange's theorem). It follows that the Sylow 3-subgroup is normalso G is not simple.

---

2) Let p be a prime and G a group of order p^r for some $r \geq 1$. Prove that, for any $s \in \mathbb{N}$ with $1 \leq s \leq r$, the group G has a subgroup of order p^s.

Proof. We shall use induction on r. The base case $r = 1$ is trivial. So suppose that $r \geq 2$ and that the result holds for groups of order p^{r-1}. Consider the center $Z(G)$ of G, which is non-trivial because $|G|$ is a prime power (a result from class). Thus, there exists $x \in Z(G)$ with $x \neq 1$. Since $|x|$ divides $|G|$ we have

$$|x| = p^n \quad \text{for some } n \in \{1, 2, ..., r\}.$$

We set $y = x^{p^{n-1}}$ so that $|y| = p$ (this is almost obvious since $y^m = x^{mp^{n-1}}$ for any $m \in \mathbb{N}$ and $|x| = p^n$ so the smallest m such that this equals the identity is p).

Let $H = \langle y \rangle$, which has order p. Notice that $H \leq Z(G)$ implies that H commutes with everything in G and so clearly $H \lhd G$. Thus, G/H is a group. In particular, from Lagrange's theorem

$$|G/H| = |G|/|H| = p^{r-1}.$$

We let $\pi : G \to G/H$ denote the canonical quotient map, i.e. $\pi(g) = gH$ for $g \in G$.

Now let $s \in \mathbb{N}$ be such that $1 \leq s \leq r$. Then $0 \leq s - 1 \leq r - 1$. By induction hypothesis, G/H has a subgroup $\overline{K}$ over order p^{s-1} (if $s - 1 = 0$ then we have the trivial subgroup), say

$$\overline{K} = \{g_1 H, g_2 H, \cdots, g_{p^{s-1}} H\}.$$

We know that $\pi^{-1}(\overline{K})$ is a subgroup of G because π is a homomorphism. But by definition

$$\begin{aligned}
\pi^{-1}(\overline{K}) &= \{g \in G \mid \pi(g) \in \overline{K}\} \\
&= \{g \in G \mid gH = g_i H \quad \text{for some } i = 1, 2, ..., p^{s-1}\} \\
&= \{g \in G \mid g \in g_i H \quad \text{for some } i = 1, 2, ..., p^{s-1}\} \\
&= g_1 H \cup g_2 H \cup \cdots \cup g_{p^{s-1}} H.
\end{aligned}$$

Since distinct cosets are disjoint, we see that

$$|\pi^{-1}(\overline{K})| = \sum_{i=1}^{p^{s-1}} |g_i H| = \sum_{i=1}^{p^{s-1}} |H| = \sum_{i=1}^{p^{s-1}} p = p \cdot p^{s-1} = p^s.$$

This shows that G has a subgroup of order p^s, which completes the proof.
