13) Find all finite groups which have exactly two conjugacy classes.

**Proof.** Let $G$ be a finite group with exactly two conjugacy classes. Since $\{1\}$ is conjugacy class, and the conjugacy classes partition $G$, the two conjugacy classes must be

$$\{1\} \text{ and } G - \{1\}.$$ 

Let $g \in G - \{1\}$. Then, by the orbit-stabilizer formula, we have

$$|G - \{1\}| = |\text{conjugacy class containing } g| = |\text{orbit of } g \text{ under the conjugation action}| = [G : \text{stab}_G(g)].$$

Since $[G : \text{stab}_G(g)]$ divides $|G|$ by Lagrange’s theorem, we deduce that

$$|G - \{1\}| \text{ divides } |G| \implies |G| - 1 \text{ divides } |G|.$$ 

This is only possible if $|G| = 2$. Thus, we have $G \simeq \mathbb{Z}/2\mathbb{Z}$. 

15) Prove that a group of order 351 has a normal Sylow $p$-subgroup for some prime $p$ dividing its order.

**Proof.** Let $G$ be a group of order $351 = 3^3 \cdot 13$. Let $n_{13}$ be the number of Sylow 13-subgroups in $G$. Then, Sylow’s Theorem Part III tells us that

$$n_{13} \text{ divides } 3^3 \text{ and } n_{13} \equiv 1 \pmod{13}.$$ 

Hence, either $n_{13} = 1$ or $n_{13} = 27$. If $n_{13} = 1$ then the unique Sylow 13-subgroup is normal and we are done. So suppose that $n_{13} = 27$ and let $P_1, ..., P_{27}$ be the distinct Sylow 13-subgroups. Observe that

(i) $P_i \cap P_j = \{1\}$ for all $i \neq j$: This is because

$$P_i \cap P_j \leq P_i \implies |P_i \cap P_j| \text{ divides } |P_i|.$$ 

But $|P_i| = 13$ is prime so $|P_i \cap P_j| = 1$ or 13. It cannot be 13 for it would imply that

$$P_i \cap P_j = P_i \implies P_i \subseteq P_j \implies P_i = P_j$$

since $P_i$ and $P_j$ have the same number of elements.

(ii) All non-identity elements in $P_i$ have order 13: This is because $P_i \simeq \mathbb{Z}/13\mathbb{Z}$ (from a previous homework) is cyclic of prime order (see also the Corollary 10 on p.90).

From these two facts we see that there are $27 \times 12 = 324$ elements of order 13. The remaining

$$351 - 324 = 27$$

elements must form a unique Sylow 3-subgroup (unique because elements in a Sylow 3-subgroup cannot have order 13 by Lagrange’s theorem). It follows that the Sylow 3-subgroup is normal and $G$ is not simple.
2) Let $p$ be a prime and $G$ a group of order $p^r$ for some $r \geq 1$. Prove that, for any $s \in \mathbb{N}$ with $1 \leq s \leq r$, the group $G$ has a subgroup of order $p^s$.

**Proof.** We shall use induction on $r$. The base case $r = 1$ is trivial. So suppose that $r \geq 2$ and that the result holds for groups of order $p^{r-1}$. Consider the center $Z(G)$ of $G$, which is non-trivial because $|G|$ is a prime power (a result from class). Thus, there exists $x \in Z(G)$ with $x \neq 1$. Since $|x|$ divides $|G|$ we have $|x| = p^n$ for some $n \in \{1, 2, ..., r\}$.

We set $y = x^{p^n-1}$ so that $|y| = p$ (this is almost obvious since $y^m = x^{mp^n-1}$ for any $m \in \mathbb{N}$ and $|x| = p^n$ so the smallest $m$ such that this equals the identity is $p$).

Let $H = \langle y \rangle$, which has order $p$. Notice that $H \leq Z(G)$ implies that $H$ commutes with everything in $G$ and so clearly $H < G$. Thus, $G/H$ is a group. In particular, from Lagrange’s theorem $|G/H| = |G|/|H| = p^{r-1}$.

We let $\pi : G \to G/H$ denote the canonical quotient map, i.e. $\pi(g) = gH$ for $g \in G$.

Now let $s \in \mathbb{N}$ be such that $1 \leq s \leq r$. Then $0 \leq s - 1 \leq r - 1$. By induction hypothesis, $G/H$ has a subgroup $K$ over order $p^{s-1}$ (if $s - 1 = 0$ then we have the trivial subgroup), say $K = \{g_1H, g_2H, \cdots, g_{p^{s-1}}H\}$.

We know that $\pi^{-1}(K)$ is a subgroup of $G$ because $\pi$ is a homomorphism. But by definition

$$\pi^{-1}(K) = \{g \in G \mid \pi(g) \in K\}$$

$$= \{g \in G \mid gH = g_iH \text{ for some } i = 1, 2, ..., p^{s-1}\}$$

$$= \{g \in G \mid g \in g_iH \text{ for some } i = 1, 2, ..., p^{s-1}\}$$

$$= g_1H \cup g_2H \cup \cdots \cup g_{p^{s-1}}H.$$ 

Since distinct cosets are disjoint, we see that

$$|\pi^{-1}(K)| = \sum_{i=1}^{p^{s-1}} |g_iH| = \sum_{i=1}^{p^{s-1}} |H| = \sum_{i=1}^{p^{s-1}} p = p \cdot p^{s-1} = p^s.$$ 

This shows that $G$ has a subgroup of order $p^s$, which completes the proof.