MATH 111A HOMEWORK 2 SOLUTIONS

---

22) If x and g are elements of the group G, prove that $|x| = |g^{-1}xg|$. Deduce that $|ab| = |ba|$.

Solution. First observe that for any $n \in \mathbb{N}$ we have

$$(g^{-1}xg)^n = \underbrace{(g^{-1}xg)\cdots(g^{-1}xg)}_{n \text{ times}} = g^{-1}x(gg^{-1})x\cdots x(gg^{-1})xg = g^{-1}x^n g.$$

Now, if $|x| = n$ is finite, then $x^n = 1$ and so $(g^{-1}xg)^n = g^{-1}x^n g = g^{-1}g = 1$. This shows that $|g^{-1}xg| \leq |x|$. Since x and g are arbitary, the same argument shows that $|x| = |g(g^{-1}xg)g^{-1}| \leq |g^{-1}xg|$. This proves that in fact $|x| = |g^{-1}xg|$. On the other hand, if $|x| = \infty$, then we must have $|g^{-1}xg| = \infty$, for otherwise

$$g^{-1}x^n g = (g^{-1}xg)^n = 1 \implies x^n = gg^{-1} = 1$$

for some $n \in \mathbb{N}$, contradicting that $|x| = \infty$. In either case, we have proved that $|x| = |g^{-1}xg|$. As a consequence, we deduce that $|ab| = |a^{-1}(ab)a| = |ba|$.

---

24) If a and b are commuting elements of G, prove that $(ab)^n = a^n b^n$ for all $n \in \mathbb{Z}$.

Solution. If $n = 0$ then clearly $(ab)^0 = 1 = a^0 b^0$. If $n \in \mathbb{Z}^+$, then we have

$$(ab)^n = \underbrace{(ab)\cdots(ab)}_{n \text{ times}} = \underbrace{(a\cdots a)}_{n \text{ times}}\underbrace{(b\cdots b)}_{n \text{ times}} = a^n b^n,$$

where we can rearrange the a's and b's since they commute. Finally, if $n \in \mathbb{Z}^-$, then

$$\begin{aligned} (ab)^n &= ((ab)^{|n|})^{-1} \\ &= ((a^{|n|}b^{|n|})^{-1} && \text{(from the above)} \\ &= b^{-|n|}a^{-|n|} && ((xy)^{-1} = y^{-1}x^{-} \text{ in general}) \\ &= \underbrace{(b^{-1}\cdots b^{-1})}_{|n| \text{ times}}\underbrace{(a^{-1}\cdots a^{-1})}_{|n| \text{ times}} \end{aligned}$$

Notice that since $ab = ba$, taking inverses on both sides, we get that $b^{-1}a^{-1} = a^{-}b^{-1}$. Hence, a^{-1} and b^{-1} commute also. So, rearranging the a^{-1}'s and b^{-1}'s above, we see that

$$(ab)^n = \underbrace{(a^{-1}\cdots a^{-1})}_{|n| \text{ times}}\underbrace{(b^{-1}\cdots b^{-1})}_{|n| \text{ times}} = a^n b^n,$$

as desired. This completes the proof.

1d) For fixed $n \in \mathbb{Z}^+$ prove that the set of rational numbers whose denominators are relatively prime to n is a subgroup under addition.

Solution. Let G be the set in question. We shall use Problem 26 from Homework1 to prove that G is a subgroup of $(\mathbb{Q}, +)$. There are three things to check.

(1) $G \neq \emptyset$: This is clear since $(n, 1) = 1$ and so $1 = \frac{1}{1} \in G$.

(2)+(3) G is closed under $+$ and inverses: Let $\frac{p}{q}, \frac{r}{s} \in G$, where $p, q, r, s \in \mathbb{Z}$ with $(q, n) = 1 = (s, n)$. Then, their sum is equal to

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + qr}{qs},$$

which lies in G because if q and s are coprime with n then so is qs (if p is a prime that divides both n and qs then it must divide either q or s also). Similarly, the inverse of $\frac{p}{q}$ is $\frac{-p}{q}$, which lies in G because $(q, n) = 1$ by assumption.

Thus, we have shown that G is a subgroup of $\mathbb{Q}$ under addition.

---

8) Let H and K be subgroups of G. Prove that $H \cup K$ is a subgroup if and only if $H \subset K$ or $K \subset H$.

Solution. First assume that $H \cup K$ is a subgroup. Suppose on the contrary that $H \not\subset K$ and $K \not\subset H$. Then, there exist $h \in H \backslash K$ and $k \in K \backslash H$. Since $h, k \in H \cup K$ and $H \cup K$ is a subgroup, we have $hk \in H \cup K$. Without loss of generality, we may assume that $hk \in H$. Now, since H is a subgroup, $h^{-1} \in H$ and so $k = (h^{-1})(hk) \in H$. This is a contradiction. Hence, either $H \subset K$ or $K \subset H$.

Conversely, assume that $H \subset K$ or $K \subset H$. Then, either $H \cup K = K$ or $H \cup K = H$. In either case $H \cup K$ is a subgroup since H and K are.

10b) Prove that the intersection of an arbitrary nonempty collection of subgroups of G is again a subgroup of G (do not assume that the collection is countable).

Solution. Let $\{H_i\}_{i \in H}$ be an arbitrary nonempty collection of subgroups of G. We shall use Problem 26 on Homework 1 to show that $H = \bigcap_{i \in H} H_i$ is a subgroup of G. There are three things to check.

(1) $H \neq \emptyset$: We have $1 \in H$ since $1 \in H_i$ for all $i \in I$ and the intersection is nonempty.

(2)+(3) H is closed under multiplicationand inverses: Let $h, k \in H$. Then $h, k \in H_i$ for all $i \in I$. Since each H_i is a subgroup, we have $hk \in H_i$ and $h^{-1} \in H_i$ for all $i \in I$. This shows that $hk \in H$ and $h^{-1} \in H$.

Therefore, indeed H is a subgroup of G.
