22) If \(x\) and \(g\) are elements of the group \(G\), prove that \(|x| = |g^{-1}xg|\). Deduce that \(|ab| = |ba|\).

**Solution.** First observe that for any \(n \in \mathbb{N}\) we have

\[
(g^{-1}xg)^n = (g^{-1}xg) \cdots (g^{-1}xg) = g^{-1}x(gg^{-1})x \cdots (g^{-1}xg) = g^{-1}x^n g.
\]

Now, if \(|x| = n\) is finite, then \(x^n = 1\) and so \((g^{-1}xg)^n = g^{-1}x^n g = g^{-1}g = 1\). This shows that \(|g^{-1}xg| \leq |x|\). Since \(x\) and \(g\) are arbitrary, the same argument shows that \(|x| = |g(g^{-1}xg)g^{-1}| \leq |g^{-1}xg|\). This proves that in fact \(|x| = |g^{-1}xg|\). On the other hand, if \(|x| = \infty\), then we must have \(|g^{-1}xg| = \infty\), for otherwise

\[
g^{-1}x^n g = (g^{-1}xg)^n = 1 \implies x^n = gg^{-1} = 1
\]

for some \(n \in \mathbb{N}\), contradicting that \(|x| = \infty\). In either case, we have proved that \(|x| = |g^{-1}xg|\). As a consequence, we deduce that \(|ab| = |a^{-1}(ab)a| = |ba|\).

24) If \(a\) and \(b\) are commuting elements of \(G\), prove that \((ab)^n = a^n b^n\) for all \(n \in \mathbb{Z}\).

**Solution.** If \(n = 0\) then clearly \((ab)^0 = 1 = a^0 b^0\). If \(n \in \mathbb{Z}^+\), then we have

\[
(ab)^n = (ab) \cdots (ab) = (a \cdots a) (b \cdots b) = a^n b^n,
\]

where we can rearrange the \(a\)'s and \(b\)'s since they commute. Finally, if \(n \in \mathbb{Z}^-\), then

\[
(ab)^n = ((ab)^{|n|})^{-1} = ((a^{|n|} b^{|n|})^{-1} = (a^{-|n|}) (b^{-|n|} = (xy)^{-1} = y^{-1} x^{-1} \text{ in general}) = (b^{-|n|} \cdots b^{-1}) (a^{-1} \cdots a^{-1})
\]

\( |n| \text{ times } |n| \text{ times } \)

Notice that since \(ab = ba\), taking inverses on both sides, we get that \(b^{-1} a^{-1} = a^{-1} b^{-1}\). Hence, \(a^{-1}\) and \(b^{-1}\) commute also. So, rearranging the \(a^{-1}\)'s and \(b^{-1}\)'s above, we see that

\[
(ab)^n = (a^{-1} \cdots a^{-1}) (b^{-1} \cdots b^{-1}) = a^n b^n,
\]

\( |n| \text{ times } |n| \text{ times } \)

as desired. This completes the proof.

1d) For fixed \(n \in \mathbb{Z}^+\) prove that the set of rational numbers whose denominators are relatively prime to \(n\) is a subgroup under addition.

**Solution.** Let \(G\) be the set in question. We shall use Problem 26 from Homework1 to prove that \(G\) is a subgroup of \((\mathbb{Q}, +)\). There are three things to check.

(1) \(G \neq \emptyset\) : This is clear since \((n, 1) = 1\) and so \(1 = \frac{1}{1} \in G\).
(2)+(3) $G$ is closed under + and inverses: Let $\frac{p}{q}, \frac{r}{s} \in G$, where $p, q, r, s \in \mathbb{Z}$ with $(q, n) = 1 = (s, n)$. Then, their sum is equal to

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + qr}{qs},$$

which lies in $G$ because if $q$ and $s$ are coprime with $n$ then so is $qs$ (if $p$ is a prime that divides both $n$ and $qs$ then it must divide either $q$ or $s$ also). Similarly, the inverse of $\frac{p}{q}$ is $-\frac{p}{q}$, which lies in $G$ because $(q, n) = 1$ by assumption.

Thus, we have shown that $G$ is a subgroup of $\mathbb{Q}$ under addition.

8) Let $H$ and $K$ be subgroups of $G$. Prove that $H \cup K$ is a subgroup if and only if $H \subseteq K$ or $K \subseteq H$.

**Solution.** First assume that $H \cup K$ is a subgroup. Suppose on the contrary that $H \not\subseteq K$ and $K \not\subseteq H$. Then, there exist $h \in H \setminus K$ and $k \in K \setminus H$. Since $h, k \in H \cup K$ and $H \cup K$ is a subgroup, we have $hk \in H \cup K$. Without loss of generality, we may assume that $hk \in H$. Now, since $H$ is a subgroup, $h^{-1} \in H$ and so $k = (h^{-1})(hk) \in H$. This is a contradiction. Hence, either $H \subseteq K$ or $K \subseteq H$.

Conversely, assume that $H \subseteq K$ or $K \subseteq H$. Then, either $H \cup K = K$ or $H \cup K = H$. In either case $H \cup K$ is a subgroup since $H$ and $K$ are.

10b) Prove that the intersection of an arbitrary nonempty collection of subgroups of $G$ is again a subgroup of $G$ (do not assume that the collection is countable).

**Solution.** Let $\{H_i\}_{i \in I}$ be an arbitrary nonempty collection of subgroups of $G$. We shall use Problem 26 on Homework 1 to show that $H = \bigcap_{i \in I} H_i$ is a subgroup of $G$. There are three things to check.

(1) $H \neq \emptyset$: We have $1 \in H$ since $1 \in H_i$ for all $i \in I$ and the intersection is nonempty.

(2) $H$ is closed under multiplication and inverses: Let $h, k \in H$. Then $h, k \in H_i$ for all $i \in I$. Since each $H_i$ is a subgroup, we have $hk \in H_i$ and $h^{-1} \in H_i$ for all $i \in I$. This shows that $hk \in H$ and $h^{-1} \in H$.

Therefore, indeed $H$ is a subgroup of $G$. 