MATH 111A HOMEWORK 4 SOLUTIONS

11b) Let F be a field and

$$G = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in F \text{ and } ac \neq 0\} \leq GL_2(F).$$

Prove that the map

$$\psi : \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto (a, c)$$

is a surjective homomorphism from G to $F^\times \times F^\times$. Describe the fibers and kernel of ψ.

Solution. First we show surjectivity. Let $(a, c) \in F^\times \times F^\times$. Then $a, c \neq 0$ implies $ac \neq 0$ and so

$$M := \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \in G.$$

Since $\psi(M) = (a, c)$ we see that ψ is surjective.

To show that it is a homomorphism, let

$$M_1 := \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \text{ and } M_2 := \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$$

be elements of G. Then, we have

$$M_1 M_2 = \begin{pmatrix} ad & ae + bf \\ 0 & cf \end{pmatrix}$$

and so $\psi(M_1 M_2) = (ad, cf) = (a, c)(d, f) = \psi(M_1)\psi(M_2)$, which shows that ψ is a homomorphism.

Finally, given any $(a, c) \in F^\times \times F^\times$, the fibre of ψ at (a, c) is

$$\psi^{-1}(a, c) = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid b \in F\}.$$

The kernel of φ is the fibre at $(1, 1)$ and so

$$\ker(\varphi) = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in F\}.$$

Extra 1) Let $\phi : G \to H$ be a group homomorphism. Prove that ϕ is an injection iff $\ker(\phi) = \{1_G\}$.

Solution. First assume that ϕ is injective. Since ϕ is a homomorphism, we know that $\phi(1_G) = 1_H$, which shows that $1_G \in \ker(\phi)$. If $g \in G$ is such that $\phi(g) = 1_H$, then we must have $g = 1_G$ by injectivity. Hence, 1_G is the only element in $\ker(\phi)$, which proves that $\ker(\phi) = \{1_G\}$.

Conversely, assume that $\ker(\phi) = \{1_G\}$. Let $g_1, g_2 \in G$ be such that $\phi(g_1) = \phi(g_2)$. Then, using the property of a homomorphism, we deduce that

$$\phi(g_1) = \phi(g_2) \implies \phi(g_1)\phi(g_2)^{-1} = 1_H \implies \phi(g_1 g_2^{-1}) = 1_H \implies g_1 g_2^{-1} \in \ker(\phi).$$

It follows that $g_1 g_2^{-1} = 1_G$ and so $g_1 = g_2$. This proves that ϕ is injective.

Extra 2) Consider $G = (\mathbb{Z}/2\mathbb{Z}, +)$ and the subgroup $H = \{-1, 1\}$ of $(\mathbb{C}^\times, \cdot)$. Show that $G \simeq H$.

Solution. Define a map $\phi : G \to H$ by

$$\phi(\overline{0}) = 1 \quad \text{and} \quad \phi(\overline{1}) = -1.$$

It is clear from the definition that ϕ is bijective. We show that ϕ is a homomorphism by direct computation:

$$\phi(\overline{0}+\overline{0}) = \phi(\overline{0}) = 1 \qquad \phi(\overline{0})\phi(\overline{0}) = 1 \cdot 1 = 1$$

$$\phi(\overline{0}+\overline{1}) = \phi(\overline{1}) = -1 \qquad \phi(\overline{0})\phi(\overline{1}) = 1 \cdot -1 = -1$$

$$\phi(\overline{1}+\overline{0}) = \phi(\overline{1}) = -1 \qquad \phi(\overline{1})\phi(\overline{0}) = -1 \cdot 1 = -1$$

$$\phi(\overline{1}+\overline{1}) = \phi(\overline{0}) = 1 \qquad \phi(\overline{1})\phi(\overline{1}) = -1 \cdot -1 = 1.$$

Hence, we have shown that ϕ is an isomorphism and so $G \simeq H$.
