MATH 111A HOMEWORK 5 SOLUTIONS

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3) Let A be an abelian group and let B be a subgroup of A. Prove that A/B is abelian. Give an example of a non-abelian group G containing a proper normal subgroup N such that G/N is abelian.

Solution. Let a_1B and a_2B be arbitrary elements in A/B, where $a_1, a_2 \in A$. Then

$$\begin{aligned}(a_1B)(a_2B) &= a_1a_2B \qquad \text{(definition of multiplication in } A/B\text{)}\\ &= a_2a_1B \qquad (A \text{ is abelian})\\ &= (a_2B)(a_1B) \qquad \text{(definition of multiplication in } A/B\text{).}\end{aligned}$$

This shows that A/B is abelian, as desired.

For the example, consider $G = D_6$, which is non-abelian. Recall that

$$D_6 = \{1, r, r^2, s, sr, sr^2\}.$$

We shall show that $K = \{1, r, r^2\}$ is a normal subgroup of D_6 by showing that it is the kernel of some group homomorphism. So define a map $\varphi : D_6 \to \mathbb{Z}/2\mathbb{Z}$ by setting

$$\varphi(s^ir^j) = \bar{i} \quad \text{for any } i, j \in \mathbb{Z}.$$

This is well-defined because $|s| = 2$. We shall show that φ is a homomorphism, i.e.

$$\varphi(xy) = \varphi(x) + \varphi(y) \quad \text{for all } x, y \in D_6.$$

We consider two cases.

(1) $x = s^ir^j$ and $y = s^kr^l$ with k even (so $s^k = 1$):

$$\begin{aligned}\varphi(s^ir^js^kr^l) &= \varphi(s^ir^{j+l}) \quad (\text{since } s^k = 1)\\ &= \bar{i}\\ &= \bar{i} + \bar{k} \quad (\bar{k} = \bar{0} \text{ since } k \text{ even})\\ &= \varphi(s^ir^j) + \varphi(s^kr^l).\end{aligned}$$

(2) $x = s^ir^j$ and $y = s^kr^l$ with k odd (so $s^k = s$):

$$\begin{aligned}\varphi(s^ir^js^kr^l) &= \varphi(s^is^kr^{-j}r^l)\\ &= \varphi(s^{i+k}r^{l-j})\\ &= \overline{i+k}\\ &= \bar{i} + \bar{k}\\ &= \varphi(s^ir^j) + \varphi(s^kr^l).\end{aligned}$$

Hence, φ is a homomorphism. Notice that

$$\ker(\varphi) = \{s^ir^j \in D_6 \mid \bar{i} = \bar{0}\} = \{1, r, r^2\} = K.$$

Hence, K is a normal subgroup of D_6 by Proposition 7 on p.82. Now, the quotient group $D_6/K = \{K, sK\}$ is clearly abelian since the identity K commutes with every element and sK commutes with itself.

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5) Use the preceding exercise (in the quotient group G/N we have $(gN)^\alpha = g^\alpha N$ for all $\alpha \in \mathbb{Z}$) to prove that the order of the element gN in G/N is n, where n is the smallest positive integer such that $g^n \in N$ (and gN has infinite order if no such positive integer exists). Give an example to show that the order of gN in G/N may be strictly smaller than the order of g in G.

Solution. For any $\alpha \in \mathbb{Z}$ we have $(gN)^\alpha = g^\alpha N$ by the previous exercise. Furthermore, notice that

$$g^\alpha N = N \iff g^\alpha \in N.$$

Hence, if $g^\alpha \notin N$ for all $\alpha \in \mathbb{N}$ then

$$(gN)^\alpha \neq N \quad \text{for all } \alpha \in \mathbb{N}$$

and so $|gN| = \infty$ (recall that N = identity of G/N). On the other hand, if there exists $\alpha \in \mathbb{N}$ with $g^\alpha \in N$, and n is the smallest as stated in the problem, then

$$(gN)^n = g^n N = N$$

and so $|gN| \leq n$. If $\alpha \in \mathbb{N}$ is such that $(gN)^\alpha = g^\alpha N = N$, then $g^\alpha \in N$ and so $\alpha \geq n$ by minimality of n. It follows that $|gN| = n$, as claimed.

For the example, let G be any non-trivial group so that it contains some element g of order ≥ 2. Take $N = G$ so that G/G is the trivial group. Consequently, gG has order 1 in G/G, which is strictly smaller than the order of g in G.

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14) Consider the additive quotient group $\mathbb{Q}/\mathbb{Z}$.

a) Show that every coset of $\mathbb{Z}$ in $\mathbb{Q}$ contains exactly one representative $q \in \mathbb{Q}$ in the range $0 \leq q < 1$.

Solution. Let $r + \mathbb{Z}$ be an arbitrary coset of $\mathbb{Z}$ in $\mathbb{Q}$, where $r \in \mathbb{Q}$. Let $[r]$ denote the integer part of r (i.e. the greatest integer that is smaller or equal to r). Let $q = r - [r]$. Then $0 \leq q < 1$ and

$$q + \mathbb{Z} = r - [r] + \mathbb{Z} = r + \mathbb{Z} \quad (\text{since } [r] \in \mathbb{Z}).$$

This shows that every coset has at least one representative $q \in \mathbb{Q}$ in the range $0 \leq q < 1$.

To show that there is at most one such representative, suppose that $p, q \in [0,1) \cap \mathbb{Q}$ and $p + \mathbb{Z} = q + \mathbb{Z}$. This implies that $p - q + \mathbb{Z} = \mathbb{Z}$ and so $p - q \in \mathbb{Z}$. But

$$0 \leq p, q < 1 \implies -1 < p - q < 1.$$

So for $p - q$ to be an integer we must have $p - q = 0$, i.e. $p = q$. Thus, we have shown that every coset of $\mathbb{Z}$ in $\mathbb{Q}$ has exactly one representative $q \in \mathbb{Q} \cap [0, 1)$.

b) Show that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order but that there are elements of arbitrarily large order.

Solution. Let $r + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$, where $r \in \mathbb{Q}$. Write $r = \frac{m}{n}$, with $m, n \in \mathbb{Z}$ and $n > 0$. Then

$$n(r + \mathbb{Z}) = m + \mathbb{Z} = \mathbb{Z} = \text{identity of } \mathbb{Q}/\mathbb{Z}$$

and so $|r + \mathbb{Z}| \leq n$. This shows that $r + \mathbb{Z}$ has finite order.

Now, for any $n \in \mathbb{N}$ consider the element $\frac{1}{n} + \mathbb{Z}$. Notice that $k(\frac{1}{n} + \mathbb{Z}) = \frac{k}{n} + \mathbb{Z}$ and

$$\frac{k}{n} + \mathbb{Z} = \mathbb{Z} \iff \frac{k}{n} \in \mathbb{Z} \quad \text{(i.e. } n \text{ divides } k).$$

Hence, the smallest positive integer k such that

$$k(\frac{1}{n} + \mathbb{Z}) = \mathbb{Z}$$

is n and so $|\frac{1}{n} + \mathbb{Z}| = n$. Since $n \in \mathbb{N}$ is arbitrary, we see that elements in $\mathbb{Q}/\mathbb{N}$ have arbitrarily large order.

22a) Prove that if H and K and normal subgroups of a group G then their intersection $H \cap K$ is also a normal subgroup of G.

Solution. Recall that by Theorem 6 on p.82 for a subgroup N in G we have

$$N \lhd G \iff gNg^{-1} \subset N \quad \text{for all } g \in G.$$

So let $g \in G$ and $x \in H \cap K$. Then $x \in H$ and $x \in K$, so

$$gxg^{-1} \in gHg^{-1} \subset H \quad \text{and} \quad gxg^{-1} \in gKg^{-1} \subset K$$

because H and K are normal in G. We deduce that

$$gxg^{-1} \in H \cap K,$$

and so $g(H \cap K)g^{-1} \subset H \cap K$, proving that $H \cap K \lhd G$.

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24) Prove that if $N \lhd G$ and H is any subgroup of G then $N \cap H \lhd H$.

Solution. By homework2 Problem 10b), we know that $N \cap H$ is a subgroup of G. Since $N \cap H \subset H$, it is a subgroup of H also. Now, to show that it is normal in H, we use the same criterion as in 22a). So let $h \in H$ and $x \in N \cap H$. Then $x \in N$ and $x \in H$, and we have

$$hxh^{-1} \in hNh^{-1} \subset N \quad \text{(since } N \lhd G \text{ and } h \in G)$$

$$hxh^{-1} \in H \quad \text{(since } h, x \in H \text{ and } H \leq G).$$

This shows that $hxh^{-1} \in N \cap H$ and so $h(N \cap H)h^{-1} \subset N \cap H$. This proves that $N \cap H$ is normal in H.
