MATH 111A HOMEWORK 8 SOLUTIONS

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14) A group H is called *finitely generated* if there is a finite set A such that $H = \langle A \rangle$.

(a) Prove that every finite group is finitely generated.

Solution. Since $G = \langle G \rangle$ trivially ($\langle A \rangle$ is the smallest subgroup in G containing A by definition), if G is finite then we can take $A = G$ to be a finite set of generators.

(b) Prove that $\mathbb{Z}$ is finitely generated.

Solution. We have $\mathbb{Z} = \langle 1 \rangle$ since by definition $\langle 1 \rangle = \{n \cdot 1 \mid n \in \mathbb{Z}\}$ (the cyclic group generated by 1).

(c) Prove that every finitely generated subgroup of the additive group $\mathbb{Q}$ is cyclic. (If H is a finitely generated subgroup of $\mathbb{Q}$, show that $H \leq \langle \frac{1}{k} \rangle$, where k is a product of all the denominators which appear in a set of generators of H.)

Solution. Suppose that $H \leq \mathbb{Q}$ is finitely generated, say

$$H = \langle \frac{n_1}{m_1}, \cdots, \frac{n_r}{m_r} \rangle, \quad \text{with } n_i, m_i \in \mathbb{Z} \text{ for all } i.$$

Let $m = m_1 \cdots m_r$. Then for each $i = 1, ..., r$ we have

$$\frac{n_i}{m_i} = (m_1 \cdots m_{i-1} m_{i+1} \cdots m_r \cdot n_i) \cdot \frac{1}{m} \in \langle \frac{1}{m} \rangle$$

since $\langle \frac{1}{m} \rangle$ contains all multiples (both positive and negative) of $\frac{1}{m}$ by definition. Since $\langle \frac{n_1}{m_1}, \cdots, , \frac{n_r}{m_r} \rangle$ is the smallest subgroup in $\mathbb{Q}$ containing all of the $\frac{n_i}{m_i}$ by definition, we deduce that

$$H = \langle \frac{n_1}{m_1}, \cdots, \frac{n_r}{m_r} \rangle \leq \langle \frac{1}{m} \rangle.$$

Since every subgroup of a cyclic group is cyclic, indeed H is cyclic.

(d) Prove that $\mathbb{Q}$ is not finitely generated.

Solution. If $\mathbb{Q}$ were finitely generated, then by part (c) it is cyclic, say

$$\mathbb{Q} = \langle \frac{1}{m} \rangle, \quad \text{where } m \in \mathbb{Z}.$$

But $\langle \frac{1}{m} \rangle = \{\frac{n}{m} \mid n \in \mathbb{Z}\}$ so elements in $\langle \frac{1}{m} \rangle$ all have denominator m. This cannot be all of $\mathbb{Q}$ because $\frac{1}{2m}$ for example cannot be written with m as its denominator. Therefore, $\mathbb{Q}$ cannot be finitely generated.

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4) Show that $S_n = \langle (12), (123 \cdots n) \rangle$ for all $n \geq 2$.

Solution. By Problem 3, S_n is generated by $\{(i, i+1) \mid 1 \leq i \leq n-1\}$. Hence, it suffices to show that $H := \langle (12), (123 \cdots n) \rangle$ contains $(i, i+1)$ for all $1 \leq i \leq n-1$. We shall use induction.

For $i = 1$ it is given that $(1, 2) \in H$. Now suppose that $(i, i+1) \in H$, with $1 \leq i \leq n-2$. From the supplementary problem below with $\sigma = (123 \cdots n)$ and $\tau = (i, i+1)$), we see that

$$(123 \cdots n)(i, i+1)(123 \cdots n)^{-1} = (i+1, i+2)$$

and so $(i+1, i+2) \in H$ also. Thus, H contains $(i, i+1)$ for all $1 \leq i \leq n-1$ and is equal to S_n.

Extra) Let $\tau = (a_1 \cdots a_m) \in S_n$ be an m-cycle ($m \leq n$) and $\sigma \in S_n$. Prove that $\sigma\tau\sigma^{-1} = (\sigma(a_1) \cdots \sigma(a_m))$.

Solution. Let $k \in \{1, ..., n\}$. We need to show that

(i) If $k \neq \sigma(a_i)$ for any $i = 1, ..., m$ then $(\sigma\tau\sigma^{-1})(k) = k$, and

(ii) If $k = \sigma(a_i)$ for some $i = 1, ..., m$ then $(\sigma\tau\sigma^{-1})(a_i) = \sigma(a_{i+1})$ (if $i - m$ we define $a_{m+1} = a_1$).

So first suppose that $k \neq \sigma(a_i)$ for any $i = 1, .., m$. This implies that $\sigma^{-1}(k) \neq a_i$ for any $i = 1, ..., m$ and so $\sigma^{-1}(k)$ is fixed by τ. Hence, we have

$$\begin{aligned} (\sigma\tau\sigma^{-1})(k) &= (\sigma\tau)(\sigma^{-1}(k)) \\ &= \sigma(\sigma^{-1}(k)) \qquad (\sigma^{-1}(k) \text{ is fixed by } \tau) \\ &= k. \end{aligned}$$

Next suppose that $k = \sigma(a_i)$ for some $i = 1, ..., m$. Then we have

$$\begin{aligned} (\sigma\tau\sigma^{-1})(k) &= (\sigma\tau)(a_i) \qquad (\sigma^{-1}(k) = a_i) \\ &= \sigma(a_{i+1}) \qquad (\tau(a_i) = a_{i+1}). \end{aligned}$$

Therefore, we have proved the claim in both case (i) and case (ii). This completes the proof.
