MATH 111A
FIRST MIDTERM
25 October 2013, 10:00–10:50

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PERM NO. ______________

PLEASE WRITE NEATLY. When giving proofs, be sure to sort out tentative ideas on scratch paper, and to put down your argument in a **logical sequence** on the test paper. The scratch work will not be graded.

1. Give clean definitions of the following concepts.

(a) A binary operation on a set G is

a map $G \times G \longrightarrow G$.

(b) A group is a pair $(G, *)$, where $*$ is a binary operation on G, $(x,y) \longmapsto *(x,y) =: x * y$, such that

(a) $(x*y)*z = x*(y*z)$ for all $x,y,z \in G$.

(b) There exists an element $e \in G$ such that $x*e = e*x = x$ for all $x \in G$.

(c) For each $x \in G$ there exists an element $x' \in G$ such that $x * x' = x' * x = e$.

(c) The order $|x|$ of an element x of a group G. Let 1 be the identity of G.

- If there is an element $n \in \mathbb{N} = \{1,2,3,\dots\}$ with $x^n = 1$, define
$|x| = \min\{n \in \mathbb{N} \mid x^n = 1\}$
- If $x^n \neq 1$ for all $n \in \mathbb{N}$, define
$|x| = \infty$.

(d) The kernel of a group homomorphism $\phi : G \to H$ is

the set
$\ker(\phi) := \{x \in G : \phi(x) = 1_H\}$.

2. In the following, let G be a group; its binary operation is written as juxtaposition as usual, unless G is specified as being additive (part **(c)**). You may use the cancellation rule for groups, but must refer to it if you do.

(a) Let $x \in G$ be an element of infinite order. Show that then $x^i \neq x^j$ whenever i, j are integers with $i \neq j$.

Let $i, j \in \mathbb{Z}$ with $i \neq j$. Since our claim is symmetric in i, j, we may assume w.l.o.g. that $i < j$, i.e. $j - i \in \mathbb{N}$. By hypothesis $x^j x^{-i} = x^{j-i} \neq 1$, and by the cancellation rule we deduce
$x^j = (x^j x^{-i}) x^i \neq 1 x^i = x^i$.

(b) Let g and x be elements of G, and suppose $|x| < \infty$. Prove that $|gxg^{-1}| = |x|$. (You may skip the easy induction involved in computing powers of gxg^{-1}.)

Say $|x| = m$. Then $x^m = 1$, and consequently
$(gxg^{-1})^m = gx^m g^{-1} = 1$.
Now let $k \in \mathbb{N}$ with $k < m$. Then $x^k \neq 1$, and the cancellation law (applied twice) allows us to deduce
$(gxg^{-1})^k = gx^k g^{-1} \neq g1g^{-1} = 1$, as required

(c) Let G be the additive group $\mathbb{Z}/36\mathbb{Z}$, and $x = \overline{-12} \in G$. Determine $|x|$. (Briefly justify your answer.)

$|x| = 3$. Indeed $x \neq \bar{0}$, $2x = -\overline{24} \neq \bar{0}$, $3x = -\overline{36} = \bar{0}$.

(d) Let $G = S_8$, the symmetric group of degree 8, and $\sigma = (1\,2\,8)(7\,4\,3\,6\,5)$. Determine $|\sigma|$ and justify your answer by indicating how to use a result (proved in class) which allows you to find the value of $|\sigma|$ at a glance. (You must state the result you use.)

$|\sigma| = \mathrm{lcm}(|(1\,2\,8)|, |(7\,4\,3\,6\,5)|) = 3 \cdot 5 = 15$,
since $|\tau| = \ell$ for any cycle τ of length ℓ, and $(1\,2\,8)$, $(7\,4\,3\,6\,5)$ are disjoint cycles.
Indeed, in class we proved: If ϱ, τ are disjoint cycles in S_n, then $|\varrho\tau| = \mathrm{lcm}(|\varrho|, |\tau|)$.

$((G,*)$ and $(H,\circ))$

3. Let ~~G and H~~ be groups and $\phi : G \to H$ an isomorphism. Moreover, denote by $\phi^{-1} : H \to G$ the inverse map of the bijection ϕ.

(a) Show that ϕ^{-1} is a group homomorphism.

Let $y_1, y_2 \in H$. Since ϕ is a surjection, there exist $x_1, x_2 \in G$ with $\phi(x_i) = y_i$ for $i = 1,2$.

In other words, $x_i = \phi^{-1}(y_i)$ for $i = 1,2$.

Then $\phi^{-1}(y_1 \circ y_2) = \phi^{-1}(\phi(x_1) \circ \phi(x_2)) \underset{\phi \text{ is a homom.}}{=}$

$\phi^{-1}(\phi(x_1 * x_2)) = x_1 * x_2 =$

$= \phi^{-1}(y_1) \circ \phi^{-1}(y_2)$

(b) Prove that H is abelian if G is abelian.

Suppose G is abelian, that is, $x_1 * x_2 = x_2 * x_1$ for all $x_1, x_2 \in G$. To see that H is abelian, let $y_1, y_2 \in H$. Again let $x_i = \phi^{-1}(y_i)$ for $i = 1,2$.

Then $y_1 \circ y_2 = \phi(x_1) \circ \phi(x_2) \underset{\text{Since } \phi \text{ is a homom.}}{=}$

$\phi(x_1 * x_2) \underset{G \text{ is abelian}}{=} \phi(x_2 * x_1)$

$\underset{\phi \text{ is a homom.}}{=} \phi(x_2) \circ \phi(x_1) = y_2 \circ y_1.$

This shows that H is abelian.

4. Let $D_{2n} = \{1, r, \ldots, r^{n-1}, s, sr, \ldots, sr^{n-1}\}$ be the dihedral group with $2n$ elements in our usual notation. Suppose that $n = 2k$, where k is an integer ≥ 2, and consider the element $z = r^k \in D_{2n}$.

Prove that $z \in Z(D_{2n})$, that is, that $zx = xz$ for all $x \in D_{2n}$. (You may use, without proof: $|r| = n$, $|s| = 2$, and $r^i s = sr^{-i}$ for $i \in \mathbb{Z}$.)

Let $x \in D_{2n}$.

If $x = r^i$ for some i, we find:

$zx = r^k r^i = r^{i+k} = r^i r^k = xz$.

Now suppose $x = sr^i$ for some $i \in \mathbb{Z}$, Then

$zx = r^k s r^i = s r^{-k} r^i = s r^{i-k}$ and

$xz = s r^i r^k = s r^{k+i}$, Since $i-k \equiv i+k$ (mod $2k$) and $|r| = n = 2k$, we thus obtain $r^{i-k} = r^{i+k}$, Hence $zx = xz$ as claimed.

Since we have covered all possibilities for x, the proof is complete.

Extra credit **5.** Show that the map $\phi : \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$, sending any residue class $\overline{x} = x + 8\mathbb{Z}$ in $\mathbb{Z}/8\mathbb{Z}$ to $\widetilde{x} = x + 4\mathbb{Z}$ in $\mathbb{Z}/4\mathbb{Z}$ is well-defined and is, in fact, a homomorphism of additive groups. Explicitly describe the kernel of ϕ.

<u>Well-definedness:</u> Suppose $\overline{x_1} = \overline{x_2} \in \mathbb{Z}/8\mathbb{Z}$, i.e. $x_1 \equiv x_2$ (mod 8). Then $x_1 \equiv x_2$ (mod 4), which shows $\widetilde{x_1} = \widetilde{x_2} \in \mathbb{Z}/4\mathbb{Z}$.

<u>ϕ is a homomorphism:</u> For $x_1, x_2 \in \mathbb{Z}$, we obtain

$$\phi(\overline{x_1} + \overline{x_2}) \overset{\text{def. of } + \text{ in } \mathbb{Z}/8\mathbb{Z}}{=} \phi(\overline{x_1 + x_2}) \overset{\text{def. of } \phi}{=} \widetilde{x_1 + x_2} \overset{\text{def. of } + \text{ in } \mathbb{Z}/4\mathbb{Z}}{=} \widetilde{x_1} + \widetilde{x_2} = \phi(\overline{x_1}) + \phi(\overline{x_2}).$$

<u>$\ker(\phi)$</u> $= \{\overline{x} \in \mathbb{Z}/8\mathbb{Z} \mid \widetilde{x} = \widetilde{0} \in \mathbb{Z}/4\mathbb{Z}\} =$

$= \{\overline{x} \in \mathbb{Z}/8\mathbb{Z} \mid x \in \mathbb{Z} \text{ with } 4 \mid x\}$

$= \{\overline{0}, \overline{4}\}$