MATH 111A
SECOND MIDTERM
20 November 2013, 10:00–10:50

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PERM NO. ______

PLEASE WRITE NEATLY. When giving proofs, be sure to sort out tentative ideas on scratch paper, and to put down your argument in a **logical sequence** on the test paper. Make sure to justify all steps. (Your scratch work will not be graded.)

1. In **(a)** – **(d)**, give clean definitions of the following concepts. Throughout, G denotes a (multiplicatively written) group.

(a) If H is a subgroup of G and $g \in G$, then

$$gH = \{gh \mid h \in H\}$$

(b) A *normal subgroup* of G. (Your definition should be in terms of cosets. You may assume that the reader knows what a subgroup is.)

A normal subgroup of G is a subgroup $N \leq G$ such that $gN = Ng$ for all $g \in G$.

(c) The *quotient group* G/N, where N is a normal subgroup of G. (You need to specify G/N as a set and define the binary operation on G/N which makes G/N into a group; but you need not check well-definedness of this operation, nor check that the group axioms are satisfied.)

$G/N = \{gN \mid g \in G\}$, with the binary operation

$G/N \times G/N \to G/N$, $(g_1N, g_2N) \mapsto g_1g_2N$

(d) Suppose H is a subgroup of G. Define the *index* $[G:H]$ ~~of H in G~~.

The index $[G:H]$ of H in G is the cardinality of the set $\{gH \mid g \in G\}$ of left cosets of H in G.

(e) Carefully state Lagrange's Theorem.

Let G be a finite group and $H \leq G$ a subgroup. Then $|G| = [G:H] \cdot |H|$.

Moreover, $[G:H]$ equals the cardinality of the set $\{Hg \mid g \in G\}$ of right cosets of H in G.

2. Suppose G is a group, $H \leq G$ a subgroup, and let $g_1, g_2 \in G$. Prove the following equivalence:

$$g_1 H = g_2 H \iff g_2^{-1} g_1 \in H.$$

"$\Longrightarrow$" Suppose $g_1 H = g_2 H$. Then $g_1 = g_1 1 \in g_1 H$ shows $g_1 \in g_2 H$; that is $g_1 = g_2 h$ for some $h \in H$. We deduce $g_2^{-1} g_1 = h \in H$ as required.

"$\Longleftarrow$" Suppose $g_2^{-1} g_1 \in H$. To show $g_1 H \subseteq g_2 H$, let $g_1 h \in g_1 H$ for some $h \in H$. Then $y := g_2^{-1} g_1 h = (g_2^{-1} g_1) h \in H$, since both $g_2^{-1} g_1$ and h belong to H, and $H \leq G$. This shows $g_1 h = g_2 y \in g_2 H$, thus proving $g_1 H \subseteq g_2 H$.
Since $H \leq G$, our hypothesis yields $g_1^{-1} g_2 = (g_2^{-1} g_1)^{-1} \in H$, and "$g_2 H \subseteq g_1 H$" follows by symmetry.

3. Let H and K be subgroups of a group G.
(a) Suppose that $K \trianglelefteq G$. Prove that then $H \cap K$ is a normal subgroup of H. (You need not prove that $H \cap K$ is a subgroup of H, but only address normality. If you use a normality criterion from class, state it.)

Normality criterion proved in class: A subgroup $N \leq G$ is normal in G iff $gNg^{-1} \subseteq N$ for all $g \in G$.

To show that $h(H \cap K)h^{-1} \subseteq H \cap K$ for all $h \in H$, let $h \in H$ and $x \in H \cap K$. Since $x \in H$ and $H \leq G$, we obtain $hxh^{-1} \in H$. Since $x \in K$ and $hKh^{-1} \subseteq K$ by normality of K, we obtain $hxh^{-1} \in K$. Thus $hxh^{-1} \in H \cap K$. Using again the normality criterion, we conclude $H \cap K \trianglelefteq H$.

(b) Extra Credit. Now suppose that both H and K are normal subgroups of G, such that $K \leq H$. Prove that the n H/K is a normal subgroup of G/K. (You need not show that H/K is a subgroup of G/K, but only address normality.)

Set $\overline{G} = G/K$, $\overline{H} = H/K$, and $\overline{g} = gK$ for $g \in G$.
We again use the normality criterion quoted under 3(a) to prove $\overline{H} \trianglelefteq \overline{G}$.
For that purpose, let $\overline{h} \in \overline{H}$ with $h \in H$, and $\overline{g} \in \overline{G}$.
Then $ghg^{-1} \in H$, since $H \trianglelefteq G$ by hypothesis, and hence

$$\overline{g} \circ \overline{h} \circ \overline{g}^{-1} \underset{\text{def. of operation on } G/H}{=} \overline{ghg^{-1}} = (ghg^{-1})K \in H/K = \overline{H}$$

as required.
Thus $\overline{g}\,\overline{H}\,\overline{g}^{-1} \subseteq \overline{H}$, whence $\overline{H} \trianglelefteq \overline{G}$ by the criterion.

4. Suppose that G and H are groups, and let $\phi : G \to H$ be a homomorphism. Prove the following implication: If $\ker(\phi) = \{1_G\}$, then ϕ is an injection.

Suppose $g_1, g_2 \in G$ with $\phi(g_1) = \phi(g_2)$.
We will show $g_2^{-1} g_1 \in \operatorname{Ker}(\phi)$: Indeed,
$\phi(g_2^{-1} g_1) = \phi(g_2^{-1}) \circ \phi(g_1) = (\phi(g_2))^{-1} \phi(g_1) =$
$= 1_H$ from $\phi(g_1) = \phi(g_2)$.
By hypothesis, this implies $g_2^{-1} g_1 = 1_G$,
and thus $g_1 = g_2$.
This shows ϕ to be an injection.

5. Suppose $G = \langle x \rangle$ is cyclic of finite order m. Use the First Isomorphism Theorem to show that $G \cong \mathbb{Z}/m\mathbb{Z}$. (You may use what we jointly proved about group elements of order m in early October, but must carefully state what you want to apply.)

From class we know: $m = |G| = |x|$, and hence $x^i = 1$ iff $i \equiv 0 \pmod{m}$, that is, $x^i = 1$ iff $i \in m\mathbb{Z}$.

Consider the map
$\phi : \mathbb{Z} \longrightarrow G$, $i \longmapsto x^i$.

1. ϕ is a homomorphism, since $\phi(i+j) = x^{i+j} = x^i x^j = \phi(i)\phi(j)$.
   power rules
2. ϕ is a surjection, because $G = \{x^i \mid i \in \mathbb{Z}\}$ by hypothesis.
3. $\ker \phi = \{i \in \mathbb{Z} \mid x^i = 1\} = m\mathbb{Z}$ by our initial remark.

In light of 1.–3., the First Isomorphism Theorem yields

$$\mathbb{Z}/\ker\phi = \mathbb{Z}/m\mathbb{Z} \cong G.$$