SELECTED SOLUTIONS TO HOMEWORK 5

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2. Prove that for each set X the topological space $(X, 2^X)$ is metrizable.

Proof. Let $d : X \times X \to \mathbb{R}$ be defined by

$$d(x,x) = 1 \quad \text{and} \quad d(x,y) = 0 \text{ if } x \neq y.$$

Then d is a metric by Exercise 7 on p. 35 (from homework 3). Let $(X, \mathcal{T}_d)$ be the topology defined by d. If we can show that $\mathcal{T}_d = 2^X$, then it means $(X, \mathcal{T}_d) = (X, 2^X)$ and so $(X, 2^X)$ is metrizable.

Clearly $\mathcal{T}_d \subset 2^X$. Conversely, let $A \in 2^X$ be a subset of X. We need to show that A is open with respect to the metric d. [The rest is an exercise. Hint: First show that $\{x\} = B(x; \delta)$ for a suitable δ so $\{x\}$ is open for all $x \in X$. Use this to show that A is open.] If you got below 10/15 on this homework and want to redo a couple of the problems to get some points back, email me (cindytsy@math.ucsb.edu) by May 19.

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4. Let $(X, \mathcal{T})$ be a topological space. Prove that $\emptyset, X$ are closed sets, that a finite union of closed sets is a closed set, and that an arbitary intersection of closed sets is a closed set.

Proof. By definition, for any subset $A \subset X$ we have

$$A \text{ is open} \iff A \in \mathcal{T} \quad \text{and} \quad A \text{ is closed} \iff C(A) \text{ is open}.$$

a) $\emptyset$ is closed: Because $\emptyset = C(X)$ and $X \in \mathcal{T}$ by O1 in Definition 2.1.

b) X is closed: Because $X = C(\emptyset)$ and $\emptyset \in \mathcal{T}$ by O2 in Definition 2.1.

c) A finite union of closed sets is closed: Let $A_1, ..., A_N$ be closed subsets in X. Then $C(A_1), ..., C(A_N)$ are open by definition. Furthermore, we have

$$C(\bigcup_{n=1}^{N} A_n) = \bigcap_{n=1}^{N} C(A_n)$$

from set theory, and this is open by O3 in Definition 2.1. So its complement

$$\bigcup_{n=1}^{N} A_n$$

is closed by definition, as desired.

d) An arbitrary intersection of closed sets is closed: Let $\{A_\alpha\}_{\alpha \in I}$ be a collection of closed subsets in X. Then $C(A_\alpha)$ is open for every $\alpha \in I$ by definition. Furthermore, we have

$$C(\bigcap_{\alpha \in I} A_\alpha) = \bigcup_{\alpha \in I} C(A_\alpha)$$

from set theory, and this is open by O4 in Definition 2.1. So its complement

$$\bigcap_{\alpha \in I} A_\alpha$$

is closed by definition, as desired.
