SELECTED SOLUTIONS TO HOMEWORK 8

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2. Let A and B be subsets of X. If A is connected, B is open and closed, and $A \cap B \neq \emptyset$, then $A \subset B$.
Solution. Consider the subset

$$P := A \cap B$$

in A. Since B is both open and closed, P is relatively open and closed in A. By assumption, $P = A \cap B \neq \emptyset$. Since A is connected, by definition, we must have $P = A$. So $A \cap B = A$, whence $A \subset B$.

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3. Let A and B be connected subsets of X. If $A \cap B \neq \emptyset$, prove that $A \cup B$ is connected.
Solution. Let $P \neq \emptyset$ be a clopen subset of $A \cup B$. To show that $A \cup B$ is connected, we must show that $P = A \cup B$. Since $P \subset A \cup B$, either $P \cap A \neq \emptyset$ or $P \cap B \neq \emptyset$. Without loss of generality, assume that $P \cap A \neq \emptyset$. Since A is connected, by Problem 2 we have $A \subset P$. Now, $A \cap B \neq \emptyset$ by hypothesis, so $B \cap P \neq \emptyset$ also. Since B is also connected, again by Problem 2 we have $B \subset P$. Therefore, $A \cup B \subset P$. But $P \subset A \cup B$ so $P = A \cup B$, as desired.

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1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Prove that f maps each interval to a single point of an interval.
Solution. Let $I \subset \mathbb{R}$ be an interval. Then I is connected by Theorem 3.4. Since f is continuous, by Theorem 2.5 $f(I)$ is connected. If $f(I)$ is a single point, then we are done. If not, then it contains two distinct points and s $f(I)$ is an interval by Theorem 2.5.

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2. Prove that a homeomorphism $f : [a, b] \to [a, b]$ carries end points into end points.
Solution. Let $c = f(a)$. Since f is surjective, we have

$$f([a, b]) = [a, b].$$

Now, because f is injective and $f(a) = c$, we have

$$f((a, b]) = [a, b]\backslash\{c\}.$$

Since $(a, b]$ is connected and f is continuous, by Theorem 2.5 $f((a, b])$ is connected also. Clearly $f((a, b])$ has more than two points because f is bijective. Hence, $f((a, b]) = [a, b]\backslash\{c\}$ must be an interval by Theorem 3.4. By Theorem 3.3, we see that $c = a$ or b. A similar argument shows that $f(b) = a$ or b. Hence, f maps endpoints to endpoints. Question: What does is mean for a subset A in X to be connected? If you are among the first three people to email me (cindytsy@math.ucsb.edu) the answer, I will give you one point back on this homework.

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2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Let $U = f(u)$ and $V = f(v)$ be such that $U \leq f(x) \leq V$ for all $x \in [a, b]$. Prove that there is a w between u and v such that $f(\omega)(b-1) = \int_a^b f(t)dt$.

Solution. Since $U \leq f(x) \leq V$ for all $x \in [a, b]$, from Calculus

$$U(b-a) = \int_a^b U dx \leq \int_a^b f(x)dx \leq \int_a^b V dx = V(b-a).$$

Hence, we have

$$f(u) = U \leq \frac{1}{b-a} \int_a^b f(x)dx \leq V = f(v).$$

It then follows from the Intermediate Value Theorem that

$$f(w) = \frac{1}{b-a} \int_a^b f(x)dx$$

for some w between u and v.