# MATH 34A WINTER 2012 FINAL REVIEW

**I. Solving equations.**

1. Solve for $x$.

$$(3x+2)^{-1} = (x-1)^{-1}$$

2. Solve for $x$.

$$(2x-1)(4x+2) = (x+3)(4x-1)$$

3. Solve for $x$ in terms of $a$.

$$ax+7 = 3x+5$$

4. Solve for $x$ in terms of $k$.

$$\frac{2}{2x-k} = \frac{3}{4x+3}$$

5. Solve for $x$ and $y$.

$$\begin{aligned} 3x - y &= 13 \\ x + y &= 7 \end{aligned}$$

6. Solve for $x$ and $y$ in terms of $a$ and $b$.

$$\begin{aligned} 3x + 2y &= 3a \\ x + y &= 2a - b \end{aligned}$$

**II. Inverse function.**

7. What is the inverse function of $f(x) = 4x - 5$? Find $f^{-1}(-1)$.
(Hint: Let $y = 4x - 5$ and try to solve for $x$ from this equation.)

8. What is the inverse function of $f(x) = x^5$? Find $f^{-1}(32)$.

9. What is the inverse function of $f(x) = 2 + x^3$. Find $f^{-1}(3)$.

**III. Error and percentage errors.**

i) error = exact value - approximate value

ii) percentage error = $\dfrac{\text{error}}{\text{exact value}} \times 100\%$

10. John thought he would only get 60/100 on his Math midterm but he actually got 85/100. What is the percentage error?

(Also see the section on tangent line approximation).

**IV. Limit.**

11. Find the limit of the following.

a) $\lim\limits_{x\to 1}(3x^2 + \sqrt{x})$ b) $\lim\limits_{x\to\infty} \frac{3}{x+1}$ c) $\lim\limits_{x\to\infty} \frac{3x-2}{x+1}$ d) $\lim\limits_{x\to -4} \frac{2x^2-32}{x+4}$

**V. Summation notation.**

12. Write out the following summations as a series of terms added together and then simplify as much as possible.

a) $\sum\limits_{n=1}^{5} n$ b) $\sum\limits_{m=-1}^{4} m^2$ c) $(\sum\limits_{k=1}^{2} x^k) \cdot (\sum\limits_{j=0}^{1} x^j)$ d) $\sum\limits_{n=0}^{2}(\sum\limits_{k=1}^{2}(n^k)$

13. Write the average of the first 150 natural numbers (including zero) using summation notation.

14. Write the sum of the squares of the first 54 positive integers using summation notation.

15. Write the sum $1 + 3 + 5 + 7 + \cdots + 45$ using a summation notation.
(Hint: Find a formula that represents the pattern and that will go in the sum.)

**VI. Proportionality.**

i) "$y$ is proportional to $x$" means "$y = kx$ for some constant $k$".

ii) "$y$ is inversely proportional to $x$" means $k \cdot \frac{1}{x}$ for some constant $k$".

16. If $y$ is inversely proportional to $x$ and $x$ is 4 when $y$ is 32 then what is $y$ when $x = 8$ and $x = 2$? Also, what is $x$ when $y = 64$?
(Hint: First use the given information to find what $k$ is.)

17. The weight of a sphere is proportional to radius cubed. If a sphere of diameter 1 cm has a mass of 2 grams, what is the diameter of the sphere that has a mass of 54 grams?

18. The time it takes to build a skyscraper is proportional to its height and inversely proportional to the number of construction workers (this means time$= k \cdot \dfrac{\text{height}}{\#\text{ of workers}}$). If it takes 10 workers 2.5 years to build a 20 story building, how long will it take 100 workers to build an 80 story building?

**VII. Linear interpolation and extrapolation.**

Given two points, $(x_1, y_1)$ and $(x_2, y_2)$ say, on a function $y = f(x)$, we find the equation of the line joining these two points and use it to approximate the function.

19. For the function $f(x) = \sqrt{x}$ we know that $f(1) = 1$ and $f(4) = 2$. Use linear interpolation to find $\sqrt{3}$.

20. The population of a city is $500,000$ in the year 2000 and it decreased to $450,000$ in the year 2005. Use linear extrapolation to estimate the population in the year 2017.

**VIII. Exponents and logarithms.**

Basic rules.

| | Exponent Rules | Logarithm Rules | Warning!!!!!!!!! |
|---|---|---|---|
| (1) | $10^a \times 10^b = 10^{a+b}$ | $\log(xy) = \log x + \log y$ | $(\log x) \cdot (\log y) \neq \log(x + y)$ |
| (2) | $10^0 = 1$ | $\log(1) = 0$ | $\log 0$ is undefined |
| (3) | $10^{-a} = \frac{1}{10^a}$ | $\log(\frac{1}{x}) = -\log x$ | – |
| (4) | $(10^a)^p = 10^{ap}$ | $\log(x^p) = p \log x$ | $\log(a \cdot x^p) \neq p \log(ax)$ |
| (5) | $\frac{10^a}{10^b} = 10^{a-b}$ | $\log(\frac{x}{y}) = \log x - \log y$ | $\frac{\log x}{\log y} \neq \log(x - y)$ |

If the unknown is in the exponents, always take log on BOTH sides of the equation and use rule (4) to bring the exponent down. Make sure you don't have zero on either side because $\log(0) =$undefined; if you have $\log(a \cdot x^p)$, use rule (1) to split it up as $\log(a) + \log(x^p)$ before applying rule (4).

21. Solve for $x$.

$$4^{x+1} = 5$$

22. Solve for $x$.

$$6^{3x-1} = 2^{x+1}$$

23. Solve for $x$.

$$3^x - 2^{x+1} = 0$$

(Hint: Move the $2^{x+1}$ to the right hand side so you don't have zero.)

**IX. Applications of exponential functions.**

i) Half life.

$$A(t) = A_0(\frac{1}{2})^{\frac{t}{K}}$$

$A(t)$ =amount of substance left after $t$ units
$A_0$ =initial amount when $t = 0$
$K$ =half life.

24. The half-life of a substance is 2 hours. At noon, there are 20 grams of it.
a) How much is there at 2pm?
b) How much is there at $5:20$pm? (Hint: What would $t$ be at $5:20$pm?)
c) When will 6 grams remain? (Use logs to solve for $t$.)

25. An element $X$ is decaying exponentially. In 1990 it had a mass of 1000 grams. But now in 2012, its mass is measured to be 20 grams only. What is its half life?

ii) Doubling time.

$$A(t) = A_0(2)^{\frac{t}{K}}$$

$A(t)$ =amount of substance left after $t$ units
$A_0$ =initial amount when $t = 0$
$K$ =doubling time.

26. A certain type of bacteria has been growing exponentially since 12AM. The doubling time is 2.5 hour.
a) How long does it take for the bacteria to triple? (Hint: You don't need $A_0$ to solve this part of the problem.)
b) If there are 1000 milligrams of it at 3AM, how much were there at 12AM?

iii) Compound interest.

$$M(n) = P_0(1 + i)^n.$$

$n$ =the number of compounding periods
$M(n)$ =amount of money after $n$ periods
$P_0$ =initial amount of money
$i = \dfrac{\text{annual interest rate}}{\text{number of compoundings per year}}$

27. Mary purchased a TV for 250 dollars using her credit card. The interest of her credit card is 15% per year, compounded every six months. If she never pays her credit card bill, how much will she owe after 5 years? (Hint: Be careful what $n$ represents in this case.)

28. Tim has $\$10,000$. Bank A offers him an annual rate of 5% but compounded yearly. Bank B offers him an annual rate of 3% but compounded monthly. If he plans to deposit his money in the bank for 10 years, which bank should he choose to get more money? (Hint: Find out how much he will get for each bank and see which value is bigger. You will have two different formulae for the two banks.)

**X. Derivatives.**

i) Definition.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

29. Use this definition to find $f'(x)$ for each of the following functions.

a) $f(x) = 3x - 5$

b) $f(x) = x^2 + 1$

c) $f(x) = \sqrt{x+1}$ (Hint: Use conjugates to simplify the expression.)

ii) Differentiation rules.

◇Sum rule:

$$[f(x) + g(x)]' = [f(x)]' + [g(x)]'$$

◇Multiplication by a constant rule:

$$[cf(x)]' = c[f(x)]'$$

◇Product rule:

$$[f(x)g(x)]' = [f(x)]'[g(x)] + [f(x)][g(x)]'$$

◇Power rule:

$$[x^n]' = nx^{n-1}$$

◇Constant rule:

$$[c]' = 0$$

◇Exponential rule:

$$[e^{Kt}]' = Ke^{Kt}$$

◇Natural log rule:

$$[\ln x]' = \frac{1}{x}$$

iii) Tangent line and tangent line approximation.

30. Let $f(x) = 2x^2 + \ln(x) + e^{-1}e^x + 1$. Find the equation of the tangent line at $x = 1$. Use it to approximate $f(2)$. Use your calculator to find the exact value and compute the percentage error. (Hint: $e^{-1}$ is just a constant.)

31. Let $f(x) = ax^3 + \sqrt{x}$. If the tangent line at $x = 1$ is parallel to the line $y = \frac{15}{2}(x-1) + 3$ what is the value of $a$? (Hint: What does it mean for two lines to be parallel?)

32. Let $f(x) = (e^{-x} + \ln x)(\frac{3}{x^2} - 2x^2)$. Find the slope of the tangent line at $x = 1$.

33. Let $f(x) = x^2$. Find the equation of tangent line at $x = a$. If $(1, 1)$ is a point on this tangent line, what is the value of $a$?

iv) Signs of first and second derivatives.

| | |
|---|---|
| $f' > 0 : f$ is increasing | $f'' > 0 : f$ is concave up |
| $f' < 0 : f$ is decreasing | $f'' < 0 : f$ is concave down |

v) Max/min problems.

At a maximum/minimum point, the derivative is zero.

34. The price of a certain computer stock $t$ days after it is issued for sale is $p(t) = 100+20t-5t^2$ dollars. The price of the stock initially rises, but eventually begins to fall. During what period of time does the stock price rise? (Hint: Notice that time cannot be negative.) To make the most profit, when should you sell it?

35. A particular is moving along the $x$-axis and its position is given by the formula $x(t) = 3t^2 - t + 1$. What is the initial position of the particle? Over what time interval is the particle traveling towards the right?

36. The height in meters of a mini rocket above ground $t$ seconds after launch is $h(t) = -t^3+12t^2$. What is the velocity of the rocket at $t = 2$? When is the velocity maximum? (Hint: To maximize velocity we need to find its derivative; don't just set it equal to zero, which will maximize/minimize the height.)

37. A farmer has 450 feet of fence and wants to build a rectangular field at the back of his house. If his house will be one side of the field (that is, he only needs to put fence on three sides

of the field), what is the maximum possible area? (Hint: Find a formula that represents the area, reduct it to have only one variable, and then differentiate.)

38. A cylindrical metal can is to have no lid. It is to have volume $8\pi\ in^3$. What height minimizes the amount of metal used (i.e. the surface area)?

39. A manufacturer sells lamps at \$6 each and sells 3000 each month. For each \$1 that the price is increased, 1000 fewer lamps are sold each month. It costs \$4 to make one lamp. What price should the lamps be sold at to maximize profit?

40.What point on the graph $y = \sqrt{x}$ is closest to $(1, 0)$? (Hint: Instead of minimizing the distance minimize the square of the distance.)

**XI. Word problems.**

Do the homework problems from Chapter 3 and Chapter 11.

**Answers.**

| | |
|---|---|
| 1. $\frac{-3}{2}$ | 21. $\frac{\log 5}{\log 4} - 1$ |
| 2. $\frac{11 \pm \sqrt{105}}{8}$ | 22. $\frac{2\log 2 + \log 3}{2\log 2 + 3\log 3}$ |
| 3. $\frac{2}{3-a}$ | 23. $\frac{-\log 2}{\log 2 - \log 3}$ |
| 4. $\frac{-3k-6}{2}$ | 24. a) 10g; b) 3.15g; c) $t \approx 3.5$ so $3:30$PM |
| 5. $x = 5, y = 2$ | 25. $\approx 3.9$ years |
| 6. $x = -a + 2b, y = 3a - 3b$ | 26. a) $\approx 4$ hours; b) $\approx 435$ milligrams |
| 7. $f^{-1}(x) = \frac{x+5}{4}; 1$ | 27. 515 dollars |
| 8. $f^{-1}(x) = x^{\frac{1}{5}}; 2$ | 28. $A = \$16289; B = \$13494$; so Bank A |
| 9. $f^{-1}(x) = (x-2)^{\frac{1}{3}}; 1$ | 29. a) 3; b) $2x$; c) $\frac{1}{2\sqrt{x+1}}$ |
| 10. $\approx 29.4\%$ | 30. $y = 6(x-1) + 4; f(2) \approx 10; 19.4\%$ |
| 11. a) 4; b) 0; c) 3; d) $-16$ | 31. $\frac{7}{3}$ |
| 12. a) 15 b) 31; c) $x + 2x^2 + x^3$; d) 8 | 32. $\frac{e-11}{e}$ |
| 13. $\frac{1}{150} \sum_{n=0}^{149} n$ | 33. $y = 2a(x-a) + a^2; 1$ |
| 14. $\sum_{n=1}^{54} n^2$ | 34. from $t = 0$ to $t = 2$; should sell when $t = 2$ |
| 15. $\sum_{n=1}^{23} (2n-1)$ | 35. $1; t > \frac{1}{6}$ |
| 16. $k = 128; 16; 64; 2$ | 36. $36\text{m/sec}; t = 4$ |
| 17. 3 cm | 37. $\frac{50625}{2}\text{ft}^2$ |
| 18. 1 year | 38. $12(\sqrt[3]{2\pi})\text{in}^2$ |
| 19. $\sqrt{3} \approx \frac{5}{3}$ | 39. \$6.5 |
| 20. $330,000$ | 49. $(\frac{1}{2}, \sqrt{\frac{1}{2}})$ |