MATH 34B WORKSHEET 3 SOLUTIONS

I. Applications of derivatives.

1. (Bacterial growth) The population $P$ of a bacterial colony $t$ days after observation begins is modeled by the quadratic function

$$P(t) = 103.5t^2 + 6900t + 230000.$$  

a) Compute and interpret the derivative of $P'(t)$.

**Solution.** $P'(t) = 103.5(2t) + 6900 = 207t + 6900$ and it represents the rate of change of population with respect to time.

b) At what rate is the population changing after 1 day? After 10 days?

**Solution.**

After 1 day: $P'(1) = 207(1) + 6900 = 7107$ units/day.

After 10 days: $P'(10) = 207(10) + 6900 = 8970$ units/day.

c) What is the initial population of the colony?

**Solution.** Initial population $P(0) = 230000$.

d) How long does it take for the population to double?

**Solution.** Double means we want the population to be 460000. Set

$$103.5t^2 + 6900t + 230000 = 460000$$

$$103.5t^2 + 6900t - 230000 = 0.$$  

Then use the quadratic formula to find $t$:

$$t = \frac{-(-6900) \pm \sqrt{(-6900)^2 - 4(103.5)(-230000)}}{2(103.5)} = -91.0684 \text{ or } 24.4017.$$  

Since time cannot be negative, it will take 24.4017 days for the population to double.

e) At what rate is the population growing at the time it doubles?

**Solution.** The rate is just the derivative at that point. So, population growing rate at this time $= P'(24.4017) = 207(24.4017) + 6900 \approx 11951$ units/day.

II. Implicit differentiation.

1. $(x^2 + 3y^2)^5 = 2xy$

**Solution.**

$$5(x^2 + 3y^2)^4(x^2 + 3y^2)' = (2x)(y)' + (2x)'(y)$$

$$5(x^2 + 3y^2)^4(2x + 6y \frac{dy}{dx}) = 2x \frac{dy}{dx} + 2y$$

$$10x(x^2 + 3y^2)^4 + 30y(x^2 + 3y^2)^4 \frac{dy}{dx} = 2x \frac{dy}{dx} + 2y$$

$$10x(x^2 + 3y^2)^4 - 2y = \frac{dy}{dx}(2x - 30y(x^2 + 3y^2)^4)$$

$$\frac{dy}{dx} = \frac{10x(x^2 + 3y^2)^4 - 2y}{(2x - 30y(x^2 + 3y^2)^4)}$$
2. \( \cos(e^y) + \cot(x^2) = 4 \)

**Solution.**

\[
\begin{align*}
\sin(e^y)(e^y)' - \csc^2(x^2)(x^2)' &= 0 \\
\sin(e^y)(e^y)\frac{dy}{dx} - 2x \csc^2(x^2) &= 0 \\
\frac{dy}{dx} &= \frac{2x \csc^2(x^2)}{e^y \sin(e^y)}
\end{align*}
\]

3. \( \ln(x^y) + \frac{1}{y} = \sec(x) \)

**Solution.** First we simplify this to get:

\[ y \ln x + y^{-1} = \sec(x). \]

Now differentiate with respect to \( x \):

\[
\begin{align*}
y(\ln x)' + (\ln x)(y)' - y^{-2}(y)' &= \sec x \tan x \\
y \cdot \frac{1}{x} + \ln x \cdot \frac{dy}{dx} - \frac{1}{y^2} \frac{dy}{dx} &= \sec x \tan x \\
\frac{dy}{dx}(\ln x - \frac{1}{y^2}) &= \sec x \tan x - \frac{y}{x} \\
\frac{dy}{dx} &= \frac{\sec x \tan x - \frac{y}{x}}{\ln x - \frac{1}{y^2}}
\end{align*}
\]

4. \( e^{\sin y} + \ln(\frac{x^2}{y}) = 1 \)

**Solution.** First we simplify this to get:

\[ e^{\sin y} + 2 \ln x - \ln y = 1. \]

Now we differentiate with respect to \( x \):

\[
\begin{align*}
(e^{\sin y})(\sin y)' + \frac{2}{x} - \frac{1}{y} \frac{dy}{dx} &= 0 \\
e^{\sin y} \cos y \cdot \frac{dy}{dx} + \frac{2}{x} - \frac{1}{y} \frac{dy}{dx} &= 0 \\
\frac{dy}{dx}(e^{\sin y} \cos y - \frac{1}{y}) &= -\frac{2}{x} \\
\frac{dy}{dx} &= \frac{-2}{x(e^{\sin y} \cos y - \frac{1}{y})}
\end{align*}
\]

**III. Related rates**

1. *(Supply rate)* When the price of a certain commodity is \( p \) dollars per unit, the manufacturer is willing to supply \( x \) hundred units, where

\[ 3p^2 - x = 12. \]

How fast is the supply changing when the price is $4 per unit and is increasing at the rate of 87 cents per month?

**Solution.** First we translate the question into math:

How fast is the supply changing (what is \( \frac{dx}{dt} \)) when the price is $4 per unit (when \( p = 4 \)) and is increasing at the rate of 87 cents per month (when \( \frac{dp}{dt} = 0.87 \) dollars/month)?
Since we want \( \frac{dx}{dt} \) at the end, we differentiate the equation with respect to \( t \):

\[
3(2p)\frac{dp}{dt} - \frac{dx}{dt} = 0.
\]

Now we plug in the given information:

\[
6(4)(0.87) - \frac{dx}{dt} = 0 \quad \Rightarrow \quad \frac{dx}{dt} = 20.88 \text{ units/month}.
\]

2. \((\text{Growth of a tumor})\) A tumor is modeled as being roughly spherical, with radius \( R \). If the radius of the tumor is currently \( R = 0.54 \text{ cm} \) and is increasing at the rate of 0.13 cm per month, what is the corresponding rate of change of the volume of the tumor? (Hint: How do you find volume of a sphere?)

**Solution.** First we translate the question into math:

If the radius of the tumor is currently \( R = 0.54 \text{ cm} \) \( (R = 0.54) \) and is increasing at the rate of 0.13 cm per month \( (\frac{dR}{dt} = 0.13\text{cm/month}) \), what is the corresponding rate of change of the volume of the tumor \( (\text{what is } \frac{dV}{dt}?) \)?

We start with

\[
V = \frac{4}{3}\pi R^3.
\]

Since we want to get \( \frac{dV}{dt} \) at the end, we differentiate this with respect to \( t \):

\[
\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3R^2 \frac{dR}{dt}.
\]

Now we plug in things we already know:

\[
\frac{dV}{dt} = 4\pi(0.54)^2(0.13) = 0.151632\pi\text{cm}^3/\text{month}.
\]