MATH 34B WORKSHEET 3 SOLUTIONS

I. Applications of derivatives.

1. (*Bacterial growth*) The population P of a bacterial colony t days after observation begins is modeled by the quadratic function

$$P(t) = 103.5t^2 + 6900t + 230000.$$

a) Compute and interpret the derivative of $P'(t)$.

Solution. $P'(t) = 103.5(2t) + 6900 = 207t + 6900$ and it represents the rate of change of population with respect to time.

b) At what rate is the population changing after 1 day? Aftet 10 days?

Solution.

After 1 day: $P'(1) = 207(1) + 6900 = 7107$ units/days.

After 10 days: $P'(10) = 207(10) + 6900 = 8970$ units/days.

c) What is the initial population of the colony?

Solution. Initial population$= P(0) = 230000$.

d) How long does it take for the population to double?

Solution. Double means we want the population to be 460000. Set

$$\begin{aligned} 103.5t^2 + 6900t + 230000 &= 460000 \\ 103.5t^2 + 6900t - 230000 &= 0. \end{aligned}$$

Then use the quadratic formula to find t :

$$t = \frac{-(6900) \pm \sqrt{(6900)^2 - 4(103.5)(-230000)}}{2(103.5)} = -91.0684 \text{ or } 24.4017.$$

Since time cannot be negative, it will take 24.4017 days for the population to double.

e) At what rate is the population growing at the time it doubles?

Solution. The rate is just the derivative at that point. So,

$$\text{population growing rate at this time} = P'(24.4017) = 207(24.4017) + 6900 \approx 11951 \text{ units/day}.$$

II. Implicit differentiation.

1. $(x^2 + 3y^2)^5 = 2xy$

Solution.

$$\begin{aligned} 5(x^2+3y^2)^4(x^2+3y^2)' &= (2x)(y)' + (2x)'(y) \\ 5(x^2+3y^2)^4(2x + 6y\frac{dy}{dx}) &= 2x\frac{dy}{dx} + 2y \\ 10x(x^2+3y^2)^4 + 30y(x^2+3y^2)^4\frac{dy}{dx} &= 2x\frac{dy}{dx} + 2y \\ 10x(x^2+3y^2)^4 - 2y &= \frac{dy}{dx}(2x - 30y(x^2+3y^2)^4) \\ \frac{dy}{dx} &= \frac{10x(x^2+3y^2)^4 - 2y}{(2x - 30y(x^2+3y^2)^4)} \end{aligned}$$

2. $\cos(e^y) + \cot(x^2) = 4$

Solution.

$$\begin{aligned} \sin(e^y)(e^y)' - \csc^2(x^2)(x^2)' &= 0 \\ \sin(e^y)(e^y)\frac{dy}{dx} - 2x\csc^2(x^2) &= 0 \\ \frac{dy}{dx} &= \frac{2x\csc^2(x^2)}{e^y\sin(e^y)} \end{aligned}$$

3. $\ln(x^y) + \frac{1}{y} = \sec(x)$

Solution. First we simplify this to get:

$$y\ln x + y^{-1} = \sec(x).$$

Now differentiate with respect to x :

$$\begin{aligned} y(\ln x)' + (\ln x)(y)' - y^{-2}(y)' &= \sec x\tan x \\ y \cdot \frac{1}{x} + \ln x \cdot \frac{dy}{dx} - \frac{1}{y^2}\frac{dy}{dx} &= \sec x\tan x \\ \frac{dy}{dx}(\ln x - \frac{1}{y^2}) &= \sec x\tan x - \frac{y}{x} \\ \frac{dy}{dx} &= \frac{\sec x\tan x - \frac{y}{x}}{\ln x - \frac{1}{y^2}} \end{aligned}$$

4. $e^{\sin y} + \ln(\frac{x^2}{y}) = 1$

Solution. First we simplify this to get:

$$e^{\sin y} + 2\ln x - \ln y = 1.$$

Now we differentiate with respect to x :

$$\begin{aligned} (e^{\sin y})(\sin y)' + \frac{2}{x} - \frac{1}{y}\frac{dy}{dx} &= 0 \\ e^{\sin y}\cos y \cdot \frac{dy}{dx} + \frac{2}{x} - \frac{1}{y}\frac{dy}{dx} &= 0 \\ \frac{dy}{dx}(e^{\sin y}\cos y - \frac{1}{y}) &= \frac{-2}{x} \\ \frac{dy}{dx} &= \frac{-2}{x(e^{\sin y}\cos y - \frac{1}{y})} \end{aligned}$$

III. Related rates

1. (*Supply rate*) When the price of a certain commodity is p dollars per unit, the manufacturer is willing to supply x hundred units, where

$$3p^2 - x = 12.$$

How fast is the supply changing when the price is \$4 per unit and is increasing at the rate of 87 cents per month?

Solution. First we translate the question into math:

How fast is the supply changing (what is $\frac{dx}{dt}$?) when the price is \$4 per unit (when $p = 4$) and is increasing at the rate of 87 cents per month (when $\frac{dp}{dt} = 0.87$ dollars/month)?

Since we want $\frac{dx}{dt}$ at the end, we differentiate the equation with respect to t :

$$3(2p)\frac{dp}{dt} - \frac{dx}{dt} = 0.$$

Now we plug in the given information:

$$\begin{aligned} 6(4)(0.87) - \frac{dx}{dt} &= 0 \\ \frac{dx}{dt} &= 20.88 \text{ units/month.} \end{aligned}$$

2. (*Growth of a tumor*) A tumor is modeled as being roughly spherical, with radius R. If the radius of the tumor is currently $R = 0.54$ cm and is increasing at the rate of 0.13 cm per month, what is the corresponding rate of change of the volume of the tumor? (Hint: How do you find volume of a sphere?)

Solution. First we translate the question into math:

If the radius of the tumor is currently $R = 0.54$ cm ($R = 0.54$) and is increasing at the rate of 0.13 cm per month ($\frac{dR}{dt} = 0.13$cm/month), what is the corresponding rate of change of the volume of the tumor (what is $\frac{dV}{dt}$?)?

We start with

$$V = \frac{4}{3}\pi R^3.$$

Since we want to get $\frac{dV}{dt}$ at the end, we differentiate this with respect to t :

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3R^2\frac{dR}{dt}.$$

Now we plug in things we already know:

$$\frac{dV}{dt} = 4\pi(0.54)^2(0.13) = 0.151632\pi \text{cm}^3/\text{month}.$$