MATH 3A FINAL REVIEW

Guidelines to taking the final exam.

1. You must show your work very clearly. You will receive no credit if we do not understand what you are doing.
2. You must cross out any incorrect work that you do not want us to grade. You can get points off for writing something wrong even if you have the correct answer.
3. You do not have to simplify your answer unless it helps you solve the problem.

Note: The materials in the last two sections (Mean Value Theorem, Growth and decay) will not appear on the final. They are here for your own interest.

I. Computing limits

When computing a limit, the first thing you should try is to just plug in the x-value. If you get a number, then that is your limit and you are done. The reason why you can do this is because most of the functions you encounter are continuous; in this case, the limit is equal to the function value at that point.

ExampleI.1. Compute

$$\lim_{x\to 1} \frac{x+2}{x^2+3}.$$

Solution. First just try plugging in $x = 1$. Then, we get

$$\lim_{x\to 1} \frac{x+2}{x^2+3} = \frac{1+2}{1^2+3} = \frac{3}{4},$$

which is a number. So this is the limit.

If you get an indeterminant form, i.e. $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then you need to do more work. There are in general three things you can do. One of them is to factor and try to cancel out the factor that is making the denominator zero. If you see a square root, then you can try to use conjugate to simplify and then cancel.

ExampleI.2. Compute

$$\lim_{x\to 3} \frac{x^2-2x-3}{x^2-9}.$$

Solution. If we plug $x = 3$, then we get

$$\frac{3^2 - 2\cdot 3 - 3}{3^2 - 9} = \frac{0}{0},$$

which is an indeterminant form. But here, we can try to factor the numerator and the denominator. If we do that, we get

$$\lim_{x\to 3} \frac{x^2-2x-3}{x^2-9} = \lim_{x\to 3} \frac{(x-3)(x+1)}{(x-3)(x+3)} = \lim_{x\to 3} \frac{x+1}{x+3}.$$

Once we cancel out the $x-3$, now we can plug in $x=3$ and get a number. Hence, the limit is

$$\lim_{x\to 3}\frac{x+1}{x+3}=\frac{3+1}{3+3}=\frac{4}{6}=\frac{2}{3}.$$

ExampleI.3. Compute

$$\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}.$$

Solution. If we plug $x=0$, then we get

$$\frac{\sqrt{4}-2}{0}=\frac{0}{0},$$

which is an indeterminant form. But there is a square root at the numerator. If we multiply both top and bottom by its conjugate, then we can simplify the limit to

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x} &= \lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}\cdot\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\\ &= \lim_{x\to 0}\frac{(x+4)-4}{x(\sqrt{x+4}+2)}\\ &= \lim_{x\to 0}\frac{x}{x(\sqrt{x+4}+2)}\\ &= \lim_{x\to 0}\frac{1}{\sqrt{x+4}+2}.\end{aligned}$$

After we cancelled out the x, now we can plug in $x=0$ and get a number. So, the limit is

$$\lim_{x\to 0}\frac{1}{\sqrt{x+4}+2}=\frac{1}{\sqrt{4}+2}=\frac{1}{4}.$$

If you don't know how to factor the polynomial or when the limit doesn't involve polynomials, then you can try to use L'Hopital's rule. Remember you can only use it when your limit is of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

ExampleI.4. Compute

$$\lim_{x\to 0}\frac{\tan 3x}{\tan 5x}.$$

Solution. If we plug in $x=0$, then we get

$$\frac{\tan 0}{\tan 0}=\frac{0}{0},$$

so we can use L'Hopital's rule. Recall that $(\tan x)'=\sec^2 x$. So, the limit is equal to

$$\lim_{x\to 0}\frac{\tan 3x}{\tan 5x}=\lim_{x\to 0}\frac{3\sec^2(3x)}{5\sec^2(5x)}.$$

Remember $\sec(x)=\frac{1}{\cos(x)}$ and $\cos(0)=1$. So plugging in $x=0$ yields

$$\lim_{x\to 0}\frac{3\sec^2(3x)}{3\sec^2(5x)}=\frac{3\sec^2(0)}{5\sec^2(0)}=\frac{3}{5}.$$

ExampleI.5. Compute

$$\lim_{x\to 0}x\ln x.$$

Solution. If we plug in $x = 0$, then we get

$$0 \ln 0 = 0 \cdot (-\infty).$$

This doesn't look like $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$. But we can rewrite the limit as

$$\lim_{x\to 0} x \ln x = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}}.$$

By writing it this way, we see that this is of the form $\frac{\pm\infty}{\pm\infty}$, so L'Hopital's rule applies. Hence, the limit is equal to

$$\begin{aligned} \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} &= \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\ &= \lim_{x\to 0} \frac{1}{x} \cdot \frac{-x^2}{1} \\ &= \lim_{x\to 0} -x. \end{aligned}$$

Now we can plug in $x = 0$ and get a number. Therefore, the limit is

$$\lim_{x\to 0} -x = -0 = 0.$$

Practice Problems.

1. Compute
$$\lim_{x\to 3} \frac{\sqrt{4+x}-1}{x^2+1}.$$
2. Compute
$$\lim_{x\to \pi} \frac{\sin(8x)}{\sin(5x)}.$$
3. Compute
$$\lim_{x\to 0} \frac{x}{\sqrt{x+1}-1}.$$
4. Compute
$$\lim_{x\to \infty} \frac{5x^4+x^2+x}{x^3-x^2+1}.$$

*5. Consider the function
$$f(x) = \begin{cases} \sqrt{x^2+a^2} & \text{if } x \le 2 \\ 2x-1 & \text{if } x > 2 \end{cases}.$$
Find the values of a for which $\lim_{x\to a} f(x)$ exists.

II. Computing derivatives

The derivative of a function is defined to be the slope of tangent at the given point and is expressed in terms of a limit. Recall that the derivative of f at the point a is the limit

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \text{ or } f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

These two definitions are equivalent. You need to know how to compute derivatives using the definition without using the differentiation rules.

ExampleII.1. Let $f(x) = x^2 + 1$. Find $f'(2)$ using the limit definition of derivative.

Solution. First we compute $f'(x)$ at an arbitrary point x. Then, by definition

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{[(x+h)^2+1]-[x^2+1]}{h}.$$

We can't just plug in $h=0$ because of the h at the denominator. We shall simplify this limit to cancel out the h so that we can plug $h=0$. We have

$$\begin{aligned}\lim_{h\to 0}\frac{[(x+h)^2+1]-[x^2+1]}{h} &= \lim_{h\to 0}\frac{[x^2+2xh+h^2+1]-[x^2+1]}{h}\\ &= \lim_{h\to 0}\frac{2xh+h^2}{h}\\ &= \lim_{h\to 0}\frac{h(2x+h)}{h}\\ &= \lim_{h\to 0}(2x+h)\\ &= 2x+0 \text{ (plug in } h=0).\end{aligned}$$

Hence, the derivative $f'(x)=2x$ and $f'(2)=2(2)=4$.

Other than the limit definition, you also need to know how to differentiate using the basic rules that you learned. You must know how to use the power rule, product rule, quotient rule, chain rule, and also the derivatives of the basisc trigonometric, exponential, and logarithmic functions.

ExampleII.2. Find $f'(x)$ for $f(x)=(\sin 3x)(2x^2+x+1)$.

Solution.

$$\begin{aligned}f'(x) &= [\sin 3x]'[2x^2+x+1]+[\sin 3x][2x^2+x+1]' \text{ (product rule)}\\ &= [3\cos 3x][2x^2+x+1]+[\sin 3x][4x+1] \text{ (chain rule, power rule).}\end{aligned}$$

ExampleII.3. Find $f'(x)$ for $f(x)=\frac{\ln(\sqrt{2x})}{\sqrt{x+1}}$.

Solution. To make this a little easier, first we can rewrite

$$\ln(\sqrt{2x})=\ln((2x)^{1/2})=\frac{1}{2}\ln(2x).$$

Then the function becomes

$$f(x)=\frac{\frac{1}{2}\ln(2x)}{\sqrt{x+1}}=\frac{1}{2}\cdot\frac{\ln(2x)}{\sqrt{x+1}}$$

Hence, the derivative is

$$\begin{aligned}f'(x) &= \frac{1}{2}\cdot(\frac{[\sqrt{x+1}][\ln(2x)]'-[\ln(2x)][\sqrt{x+1}]'}{[\sqrt{x+1}]^2}) \text{ (quotient rule)}\\ &= \frac{1}{2}\cdot\frac{\sqrt{x+1}(\frac{1}{2x}\cdot 2)-\ln(2x)(\frac{1}{2\sqrt{x+1}})}{x+1}\\ &= \frac{1}{2}\frac{\sqrt{x+1}-\frac{\ln(2x)}{2\sqrt{x+1}}}{x+1}.\end{aligned}$$

You can also simplify this to

$$f'(x)=\frac{1}{2}\frac{\frac{2(x+1)-\ln(2x)}{2\sqrt{x+1}}}{x+1}=\frac{1}{4}\frac{2x+2-\ln(2x)}{(x+1)^{3/2}}.$$

Practice Problems.

1. Find $f'(x)$ for $f(x) = \sqrt{2x+1}$ using the limit definition of derivative.

2.Use your memory, textbook, or the internet, find the derivative of the following functions.

$$\sin x, \cos x, \tan x, \sec x, e^x, 2^x, 3^x, \ln x, \sqrt{x}.$$

3. Find the derivatives of the following functions.

a) $\cos(2x)$
b) e^{x^2}
c) $\sqrt{4x^2-1}$

You should be able to do these with no difficulties.

4. Find $f'(x)$ for

$$f(x) = \frac{1}{\sqrt{2x^2+\sin(9x)}}.$$

5. Find $f'(x)$ for

$$f(x) = e^{4x^2}\sin(\ln(2x)).$$

6. Find $f'(x)$ for

$$f(x) = \ln(\ln(\sin(x^2))).$$

*7. Find $f'(x)$ for

$$f(x) = \begin{cases} x^2\sin(\frac{1}{x}) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}.$$

III. Implicit differentiation

When y isn't given as an explicit function of x or when it is impossible to solve y explicitly in terms of x, e.g. $y^2 + \sin y = x$, then you would need to use implicit differentiation to find the derivative. The main thing to keep in mind is that $y = f(x)$ depends on x and is implicity a function of x. So when you take the derivative, you would need to use the chain rule, e.g. $\frac{d}{dx}(y^2) = 2y \cdot y'$.

ExampleIII.1. Find y' if $2\sqrt{x} + \sqrt{y} = x$.

Solution. First differentiate the whole equation with respect to x. We get

$$\begin{aligned} [2x^{\frac{1}{2}} + y^{\frac{1}{2}}]' &= [x]' \\ 2 \cdot \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}} \cdot y' &= 1 \\ \frac{1}{\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot y' &= 1. \end{aligned}$$

Now we just solve for y' to get

$$y' = (1 - \frac{1}{\sqrt{x}}) \cdot 2\sqrt{y}.$$

ExampleIII.2. Find y' at the point $(0, 1)$ if $1 + xy^2 - y^2 = x$.

Solution. Again first differentiate the whole equation with respect to x. We get

$$\begin{aligned}[1+xy^2-y^2]' &= [x]' \\ (y^2 - x\cdot 2yy') - 2yy' &= 1 \text{ (product rule)}.\end{aligned}$$

Now we isolate y' to get

$$\begin{aligned}y^2 - 2xyy' - 2yy' &= 1 \\ y^2 - 1 &= y'(2xy+2y) \\ y' &= \frac{y^2-1}{2xy+2y}.\end{aligned}$$

To find y' at the point $(0,1)$, we simply plug in $x=0$ and $y=1$. Hence, we have

$$y'|_{x=0,y=1} = \frac{1-1}{0+2} = 0.$$

Practice Problems.

1. Find y' at the point $(1,1)$ if $x^2+xy+y^2=3$.
2. Find y' if $x^2y^2 + x\sin y = 4$.

IV. Related rates

One application of implicit differentiation is related rates. The idea is, sometimes in a problem, there are multiple variables that are related to each other. We want to write down an equation that relates them together and differentiate it to find how their derivatives are related. Usually you will be given the rate of change of some of the variables. Using the given information, you can find out the rate of change of another variable which is not given. Pictures really help in doing related rates problems.

ExampleIV.1. Gravel is being dumped from a conveyor belt at a rate of 30ft^3/min, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft?

Solution. First volume of a cone is given by the formula

$$V = \frac{1}{3}(\text{base area})(\text{height}) = \frac{1}{3}\pi r^2 h.$$

We know that the diameter $2r$ is equal to the height h, so this simplifies to

$$V = \frac{1}{3}\pi r^2(2r) = \frac{2}{3}\pi r^3.$$

Differentiating both sides with respect to t yields

$$\frac{dV}{dt} = \frac{2}{3}\pi(3r^2\frac{dr}{dt}) = 2\pi r^2\frac{dr}{dt}.$$

We know that $\frac{dV}{dt}=30$ and we are looking at the instance when $h=10$, i.e. $r=5$. Plugging them into the equation gives

$$\begin{aligned}30 &= 2\pi(25)\frac{dr}{dt} \\ \frac{dr}{dt} &= \frac{30}{50\pi} = \frac{3}{5\pi}.\end{aligned}$$

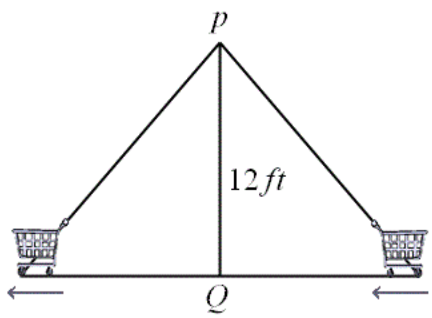
Practice Problems.

1. A cylindrical tank with radius 5 m is being filled with water at a rate of 3 m^3/min. How fast is the height of the water increasing?

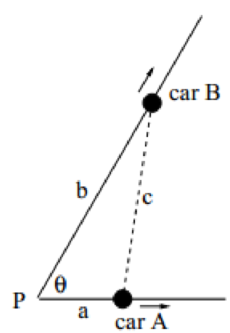
2. A balloon leaves the ground 500 feet away from a observer, and rises vertically at the rate of 140 feet per minute. At what rate is the angle of inclination of the observer's line of sight increasing at the instant when the balloon is at an altitude 500 feet higher than the observer's head?

3. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of the string has been let out?

*4. Two carts, A and B, are connected by a rope 39 ft long that passes over a pulley P. The point Q is on the floor 12 ft directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 2ft/s. How fast is cart B moving toward Q at the instant when A is 5ft from Q?



*5. Two straight roads intersect at point P at 60° angle. Car A is traveling away from P on one road and car B on the other road. You are in car A, which has a device that can measure distance from car B and also the rate at which that distance is increasing. At a certain moment, you have traveled 3 km away from P and are moving at 80 km/hr. At that time the device shows that your distance from car B is 7 km, and this distance is increasing at 100 km/hr. Find the speed at which car B is moving at this instance.



V. Linear approximation

One application of derivative is linear approximation. The idea is, if you know a point on a function, you can use derivative to find the equation of tangent at that point. Then, instead of plugging in an x-value into the function (which might be impossible to do by hand), we can plug it into the equation of the tangent line instead. If the x-value is close to the base point of the tangent line, then the approximation would be quite close to the actual value.

ExampleV.1. Use a linear approximation to estimate $\sqrt[4]{80}$. Is the approximation bigger or smaller than the actual value?

Solution. Since we are trying to estimate a 4th root, let us use the function $f(x) = x^{\frac{1}{4}}$. We also know that $f(81) = \sqrt[4]{81} = 3$ and 81 is close to 80, so let us use the point $(81, 3)$ as our base point. So, the equation of tangent at $x = 81$ is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= m(x - 81). \end{aligned}$$

To find m, which is the derivative of $f(x)$ at $x = 81$, first we compute $f'(x)$. Using power rule, we have

$$f'(x) = \frac{1}{4}x^{\frac{-3}{4}} = \frac{1}{4x^{\frac{3}{4}}}.$$

Plugging in $x = 81$ yields

$$m = \frac{1}{4(81)^{\frac{3}{4}}}.$$

Remember, raising to the $\frac{3}{4}$th power is the same is first taking the 4th root, and then cubing the result. Hence, we can simplify this to

$$m = \frac{1}{4(3)^3} = \frac{1}{4(27)} = \frac{1}{108}.$$

Therefore, the equation of tangent at $x = 81$ is

$$y - 3 = \frac{1}{108}(x - 81).$$

Now to estimate $\sqrt[4]{80}$, which corresponds to $x = 80$ on the function $f(x) = \sqrt[4]{x}$, we simply plug $x = 80$ into the equation above and solve for y. Hence, we get

$$\sqrt[4]{80} \approx \frac{1}{108}(80 - 81) + 3 = \frac{-1}{108} + 3.$$

Finally, our approximation $\frac{-1}{108} + 3$ is bigger than the actual value $\sqrt[4]{80}$ because the graph of $\sqrt[4]{x}$ is concave down (any function x to a power between 0 and 1 is concave up). So if we draw the tangent line, it would lie above the function. Consequently, the approximation is bigger.

Practice Problems.

1. Use a linear approximation to estimate $\sqrt[5]{31}$. Is your approximation bigger or smaller than the actual value?

2. Use a linear approximation to $e^{0.002.}$ Is your approximation bigger or smaller than the actual value?

3. Water is being pumped into a tank at a decreasing rate (i.e. the rate is getting slower and slower). If initially there is 1000m^3 of water in the tank, and the rate at which water is being pumped in is $200\text{m}^3/\text{min}$, use linear approximation to estimate the amount of water in the tank after 5 minutes. Is your approximation bigger or smaller than the actual value?

VI. First and second derivatives

The first and second derivatives tell us the shape of the graph. Remember that

$$f' > 0 \iff f \text{ is increasing}$$

$$f' < 0 \iff f \text{ is decreasing}$$

$$f'' > 0 \iff f \text{ is concanve up}$$

$$f'' < 0 \iff f \text{ is concave down.}$$

You should understand why these are true. Also, at points where we have a local maximum or local minimum, the tangent is horizontal and so the derivative is 0 (it is also possible that there is no tangent line and the derivative is undefined). Hence, to find local extrema of a function, we simply have to find all the critical points and test whether they are local maximum, local minimum, or neither.

There are two ways to test the critical points. One way is to look at the first derivative to see how the sign changes at the point. For example, if it goes from $-$ to $+$, i.e. the original function goes from decreasing to increasing, then the point is obviously a local minimum. Another way is to look at the second derivative. For example, if it is $+$, i.e. the original function is concave up, then the point is a local minimum.

ExampleVI.1. Consider the function $f(x) = x^5 - 5x + 3$.

a) Find the intervals on which f is increasing or decreasing.
b) Find the local maximum and minimum values of f.
c) Find the intervals of concavity and the inflection points.

Solution.

a) First the derivative of $f(x)$ is

$$f'(x) = 5x^4 - 5 = 5(x^4 - 1).$$

To find the critical points, we set this equal to zero. We get

$$\begin{aligned} 5(x^4 - 1) &= 0 \\ x^4 - 1 &= 0 \\ x^4 &= 1 \\ x &= \pm 1. \end{aligned}$$

It is not hard to see that $f'(x) = 5(x^4 - 1)$ is positive when $|x| > 1$ and negative when $|x| < 1$. Hence, we have the following table.

	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$f'(x)$	$+$	0	$-$	0	$+$

Therefore, f is increasing on the intervals $(-\infty, -1)$ and $(1, \infty)$, and is decreasing on the interval $(-1, 1)$.

b) Local extrema occur at the critical points of a funciton. From part a), there are only two critical points, namely $x = -1$ and $x = 1$. Since $f'(x)$ changes from $+$ to $-$ at $x = -1$, i.e. the function $f(x)$ first increases then decreases, we see that $x = -1$ is a local maximum. Similarly, the point $x = 1$ is a local minimum.

c) Second derivative tells us about concavity. We have

$$f''(x) = 20x^3,$$

which has a zero at $x = 0$. Also, we have the following table.

	$x < 0$	$x = 0$	$x > 0$
$f''(x)$	$-$	0	$+$

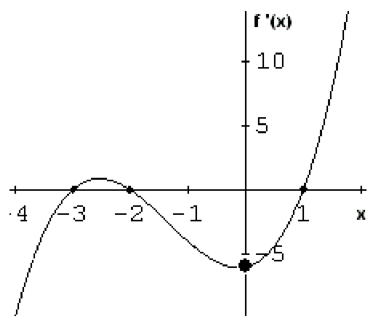
Therefore, f is concave up on the interval $(0, \infty)$ and is concave down on the interval $(-\infty, 0)$. Inflections points are points where the second derivative changes sign, i.e. points at which the concavity changes. Hence, $x = 0$ is an inflection point.

Practice Problems.

1. Consider the function $f(x) = 2x^3 + 3x^2 - 36x$.

a) Find the intervals on which f is increasing or decreasing.
b) Find the local maximum and minimum values of f.
c) Find the intervals of concavity and the inflection points.
d) Sketch the graph of $f(x)$ using parts a) through c). Label the x- and y-intercepts and the coordinates of all local extrema.

2. Below shows the graph of $f'(x)$. Use it to answer parts a) through c) as in Problem1.



VII. Optimization

We can use derivatives to optimize a function, i.e. find its absolute maximum or absolute minimum. The important thing to understand is the absolute maximum or minimum occurs at the critical points of a function. When you are looking at a function on a closed interval, then they can occur at the endpoints also. Hence, to find the absolute extrema of a function on a closed interval, all you have to do is to find all the critical points and the endpoints. After that, you simply plug these x-values into the function to see which one gives you the biggest number, and which one gives you the smallest.

In terms of solving a word problem on optimization, the first thing you do is always write down an expression that represents the thing you are trying to optimize. After that, you need to use the given condition to reduce the expression to have only one variable. Once there is only one variable, you can take the derivative to find all the critical points, and then you just check if which gives you a maximum, and which gives you a minimum (if there are endpoints, you have to check them also). Pictures also help in doing optimization problems.

ExampleVII.1. Find the absolute maximum and absolute minimum of the function $f(x) = 2x^3 - 3x^2 - 12x$ on the interval $[0, 3]$.

Solution. First we find the critical points of the function. The derivative is

$$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$$

and setting it equal to 0 yields

$$\begin{aligned} 6(x^2 - x - 2) &= 0 \\ 6(x-2)(x+1) &= 0 \\ x &= 2 \text{ or } -1. \end{aligned}$$

The point $x = -1$ is outside of the interval $[0, 3]$ so we can ignore it. So now we have three points where the absolute maximum and minimum can occur, namely the endpoints $x = 0, x = 3$, and the critical poitn $x = 2$. We plug them back into the original function to test which one is biggest and which one is smallest. We see that

$$\begin{aligned} f(0) &= 0 \\ f(3) &= 2(27) - 3(9) - 36 = -9 \\ f(2) &= 2(8) - 3(4) - 24 = -20. \end{aligned}$$

Hence, the absolute maximum is $y = 0$ which occurs at $x = 0$, and the absolute minimum is $y = -20$ which occurs at $x = 2$.

ExampleVII.2. A box with a square base and open top must have a volume of $32,000$ cm^3. Find the dimensions of the box that minimizes the amount of material used.

Solution. We want to minimize the amount of material used, i.e. minimize the surface area of the box. Since the box has no top, it has a base and four sides and the surface area is given by

$$\begin{aligned} S &= \text{(base area)} + 4\text{(side area)} \\ &= s^2 + 4sh. \end{aligned}$$

We need to get this down to one variable. The only other thing we are given is the volume, which is $32,000$. But we know how to find the volume in terms of s and h. Hence, we have the relation

$$32,000 = \text{(base area)} \times \text{(height)} = s^2 h.$$

Dividing both sides by s^2 yields

$$h = \frac{32000}{s^2}.$$

Now plug this back into the formula for S. We get

$$S = s^2 + 4s \cdot \frac{32000}{s^2} = s^2 + \frac{128000}{s}.$$

To find the minimum, we take the derivative

$$S' = 2s - \frac{128000}{s^2}$$

and set it equal to 0. We then get

$$\begin{aligned} 2s - \frac{128000}{s^2} &= 0 \\ 2s &= \frac{128000}{s^2} \\ s^3 &= 64000 \\ s &= \sqrt[3]{64000} = 40. \end{aligned}$$

To see that this really gives us a minimum, we can use the second derivative test. The second derivative is

$$S'' = 2 + 2 \cdot \frac{128000}{s^3},$$

which is always positive for $s > 0$ (length must be nonnegative). This means the function is always concave up, and so the critical point $s = 40$ must be a minimum. Plugging it back into the formula for h, we see that

$$h = \frac{32000}{40^2} = \frac{32000}{1600} = 20.$$

Therefore, the dimensions that will minimize the material used is $s = 40$ and $h = 20$.

ExampleVII.3. Find the area of the largest rectangle that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 8 - x^2$.

Solution. We want to maximize the area of a rectangle, which is given by the formula

$$A = wl.$$

Draw a picture for this problem, and you will see that the width is 2 times the x-coordinate of the vertex that lies on the parabola, and the height is just the y-coordinate, which we know is $y = 8 - x^2$. Hence, the area is

$$A = (2x)(8 - x^3) = 16x - 2x^4.$$

To maximize this, we take the derivative

$$A' = 16 - 8x^3 = 8(2 - x^3)$$

and set is equal to 0. We get

$$\begin{aligned} 8(2 - x^3) &= 0 \\ 2 - x^3 &= 0 \\ x^3 &= 2 \\ x &= \sqrt[3]{2}. \end{aligned}$$

To check that this gives a maximum, we use the second derivative test. The second derivative is

$$A'' = -24x^2,$$

which is always negative. This means the function is always concave down, so the critical point $x = \sqrt[3]{2}$ must give us a maximum. To get the actual maximum area, we plug this back to the area function. We get that

$$A(\sqrt[3]{2}) = 16(\sqrt[3]{2}) - 2(\sqrt[3]{2})^4.$$

Practice Problems.

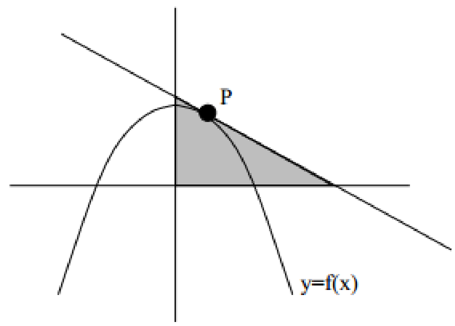
1. Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

2. If 1200cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

3. A rectangular storage container with an open top is to have a volume of $10\ \text{m}^3$. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.

*4. A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

**5. Consider the graph of $f(x) = 1 - x^2$ and a typical point P on the graph in the first quadrant. The tangent line to the graph at P will determine a right triangle in the first quadrant, as pictured below. Find P so that the area of the triangle is as small as possible



VIII. Mean Value Theorem

Recall that the Mean Value Theorem says that if a function $f(x)$ is differentiable on a closed interval $[a, b]$, then the average rate of change, i.e. slope of the secant line, is always attained. In other words, there always exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

If $f(x)$ has the same value at the two endpoints, i.e. $f(a) = f(b)$, then this is just Rolle's Theorem, and it says that the derivative has to vanish somewhere between a and b. These two theorems are extremely useful.

ExampleVIII.1. For the function $f(x) = x^3 - 2x + 1$ on the interval $[-2, 3]$, find all numbers c that satisfy the conclusion of the Mean Value Theorem.

Solution. The average rate of change of $f(x)$ on the interval $[-2, 3]$ is

$$\frac{f(3) - f(-2)}{3 - (-2)} = \frac{(27 - 6 + 1) - (-8 + 4 + 1)}{5} = \frac{22 + 3}{5} = 5.$$

Hence, the c that satisfies the conclusion of the Mean Value Theorem is a number such that

$$f'(c) = 5.$$

Now, the derivative is

$$f'(x) = 3x^2 - 2.$$

So we are just solving the equation

$$\begin{aligned} 3c^2 - 2 &= 5 \\ 3c^2 &= 7 \\ c^2 &= \frac{7}{3} \\ c &= \pm\sqrt{\frac{7}{3}}. \end{aligned}$$

Notice that $\frac{7}{3}$ is a little bit more than 2, so its square is a little bit more than 1. Hence, both $\pm\sqrt{\frac{7}{3}}$ lie within the interval $[-2,3]$. So $c=\sqrt{\frac{7}{3}}$ and $c=-\sqrt{\frac{7}{3}}$ are two numbers that satisfy the conclusion of the Mean Value Theorem in this case.

ExampleVIII.2. Show that the equation $x^3-3x^2+3x+10=0$ has at most two roots.

Solution. Let $f(x)=x^3-3x^2+3x+10$. If it has three or more zeroes, then by the Mean Value Theorem (or Rolle's Theorem), the derivative would have to vanish at least two times. But the derivative

$$f'(x)=3x^2-6x-3=3(x^2-2x+1)=3(x-1)^2$$

has only one root. So there can at most be two zeros.

Practice Problems.

1. For the function $f(x)=x^2-4x+5$ on the interval $[1,5]$, find all numbers c that satisfy the conclusion of the Mean Value Theorem.

2. For the function $f(x)=\sqrt{x+2}$ on the interval $[-1,2]$, find all numbers c that satisfy the conclusion of the Mean Value Theorem.

3. Show that the equation $x^5-6x+c=0$ has at most one root in the interval $[-1,1]$.

*4. Suppose f is differentiable on $[2,5]$ and $1\le f'(x)\le 4$ for all x in $(2,5)$. Show that $3\le f(5)-f(2)\le 12$.

IX. Growth and decay

Recall that the natural model for growth or decay of a function $y(t)$ (it usually represents population or the amount of a certain thing at time t) is

$$\frac{dy}{dt}=ky,\ \text{where } k>0 \text{ if it is growth and } k<0 \text{ if it is decay.}$$

This differential equation simply says that the rate of change is proportional to the amount present. The general solution to this differential equation is

$$y=y_0e^{kt},\ \text{where } y_0=y(0) \text{ is the initial amount.}$$

A special case of this is the cooling model. If $T(t)$ represents the temperature at time t, then Newton's law of cooling says that

$$\frac{dT}{dt}=k(T-T_s),\ \text{where } T_s \text{ is the surrounding temperature.}$$

This simply says that the rate of change of temperature is proportional to its difference with the surrounding temperature, which makes sense. To see what the general solution is, observe that this can be realized as a special case of the growth/decay model above. Notice that subtracting the constant T_s from T will not change its derivative. So we have

$$\frac{d(T-T_s)}{dt}=k(T-T_s).$$

Viewing the whole thing $T-T_s$ as the function y in the decay model, the general solution is then

$$T-T_s=(T_0-T_s)e^{kt},\ \text{i.e. } T=(T_0-T_s)e^{kt}+T_s,$$

where $T_0=T(0)$ is the initial temperature.

ExampleIX.1. A sample of tritium-3 decayed to 94.5% of its original amount after a year.

a) What is the half-life of tritium-3?
b) How long would it take the sample to decay to 20% of its original amount?

Solution. Let $y(t)$ denote the amount of the sample after t years.

a) This is a decay model problem and we know that the general solution of $y(t)$ is

$$y(t) = y_0 e^{kt}, \text{ where } y_0 = y(0) \text{ is the initial amount.}$$

We are given that only 94.5% of the sample is left after 1 year, that is,

$$y(1) = 94.5\% \text{ of the original} = (94.5\%)y_0 = 0.945y_0.$$

This gives us an equation to solve for k. We have

$$\begin{aligned} 0.945y_0 &= y_0e^{k(1)} \\ 0.945 &= e^k \text{ (the } y_0 \text{ cancels)} \\ \ln 0.945 &= k. \end{aligned}$$

Plugging it back to $y(t)$ tells us that the solution is

$$y(t) = y_0 e^{(\ln 0.945)t}.$$

To find the half-life means to find how long it takes for the sample to decrease to half of its original size, i.e. we want to solve for t in the equation

$$\begin{aligned} y(t) &= 0.5y_0 \\ y_0e^{(\ln 0.945)t} &= 0.5y_0 \\ e^{(\ln 0.945)t} &= 0.5 \text{ (the } y_0 \text{ cancels)} \\ (\ln 0.945)t &= \ln 0.5 \\ t &= \frac{\ln 0.5}{\ln 0.945}. \end{aligned}$$

Thus, the half-life is $\frac{\ln 0.5}{\ln 0.945}$ years.

b) To find out how long it takes for the sample to decrease to 20% of its original amount means we want to solve for t in the equation

$$\begin{aligned} y(t) &= (20\%)y_0 \\ y_0e^{(\ln 0.945)t} &= 0.2y_0 \\ e^{(\ln 0.945)t} &= 0.2 \\ (\ln 0.945)t &= \ln 0.2 \\ t &= \frac{\ln 0.2}{\ln 0.945}. \end{aligned}$$

ExampleIX.2. Professor Farlow always has a cup of coffee before his 8 : 00AM class. Suppose the coffee is 200°F when poured from the coffee pot at 7 : 30AM, and 15 minutes later it cools to 120°F in a room whose temperature is 70°F. However, Professor Farlow never drinks his coffee until it cools to 90°F. When will the professor be able to drink his coffee?

Solution. Let $T(t)$ be the temperature of the coffee t minutes after 7 : 30AM (it is important to specify the units for time and the starting point, i.e. what time $t = 0$ corresponds to, otherwise you will get very confused). Then, the given information can be expressed as

$$T(0) = 200, \; T(15) = 120, \; T_s = 70,$$

and we want to find out for what t will $T(t) = 90$. We know that the solution for the function $T(t)$ is

$$T(t) = (200 - 70)e^{kt} + 70 = 130e^{kt} + 70.$$

To find the cooling constant k, we use the only information we haven't used, i.e. $T(15) = 120$. We get

$$\begin{aligned} 120 &= 130e^{k(15)} + 70 \\ 50 &= 130e^{15k} \\ e^{15k} &= \frac{5}{13} \\ 15k &= \ln\frac{5}{13} \\ k &= \frac{1}{15}\ln\frac{5}{13}. \end{aligned}$$

Plugging it back into the formula yields

$$T(t) = 130e^{(\frac{1}{15}\ln\frac{5}{13})t} + 70.$$

Now we just need to solve when this is equal to 90. We have

$$\begin{aligned} 130e^{(\frac{1}{15}\ln\frac{5}{13})t} + 70 &= 90 \\ 130e^{(\frac{1}{15}\ln\frac{5}{13})t} &= 20 \\ e^{(\frac{1}{15}\ln\frac{5}{13})t} &= \frac{2}{13} \\ (\frac{1}{15}\ln\frac{5}{13})t &= \ln\frac{2}{13} \\ t &= \frac{\ln\frac{2}{13}}{\frac{1}{15}\ln\frac{5}{13}}. \end{aligned}$$

Hence, Professor Farlow should drink his coffee $(\ln\frac{2}{13})/(\frac{1}{15}\ln\frac{5}{13})$ minutes after 7 : 30AM.

Practice Problems.

1. A bacteria culture grows at a rate proportional to its size. After 2 hours there are 600 bacteria and after 8 hours the count is $75,000$.

a) Find the initial population.
b) Find an expression for the population after t hours
c) Find the number of cells after 5 hours.
d) Find the rate of growth after 5 hours.
e) When will the population reach $200,000$?

2. A certain colony of bacteria grows at a rate proportional to the number of bacteria present. Suppose the number of bacteria doubles every 12 hours. How long (in hours) will it take this colony to grow to five times its original size?

3. A thermometer is taken from a room where the temperature is $20°C$ to the outdoors, where the temperature is $5°C$. After one minute the thermometer reads $12°C$.

a) What will the reading on the thermometer be after t minutes? What about after 2 minutes?
b) When will the thermometer read $6°C$?
c) Intuitively, what should the temperature tend to as $t \to \infty$? Verify that the model makes sense by checking that the function you obtained in part a) tends to your guess as $t \to \infty$.

4. In a murder investigation, a corpse was found by a detective at exactly 8 PM Being alert, he measures the temperature of the body and finds it to be 70°F. Two hours later the detective again measures the temperature of the corpse and finds it to be 60°F. If the room temperature is 50°F, and assuming the body temperature of the person before death was 98.6°F, how many hours before 8 PM did the murder occur?