MATH 3C EXAM 2 REVIEW PROBLEMS PART 1

Example (Undetermined coefficients)**.** Solve the differential equation

$$y' - y = 2t + 3$$

Step1: Solve for a homogeneous solution y_h.

We want to find a solution to the corresponding homogeneous differential equation $y' - y = 0$. We know that the solution is of the form $y_h = e^{rt}$. Plugging in

$$y_h' = re^{rt} \text{ and } y_h = e^{rt}$$

into the equation, we get that

$$\begin{aligned} re^{rt} - e^{rt} &= 0 \\ e^{rt}(r-1) &= 0 \\ r - 1 &= 0. \end{aligned}$$

Hence, $r = 1$ and so $y_h = e^t$ is a homogeneous solution. (Alternatively, you can also use separation of variables to find y_h.)

Step2: Solve for a particular non-homogeneous solution y_p.

We want to find a solution to the original differential equation $y' - y = 2t + 3$. Since the forcing function $f(t) = 2t + 3$ is a linear function, we guess that $y_p = at + b$ is also a linear function. Plugging in

$$y_p' = a \text{ and } y_p = at + b$$

into the equation, we get that

$$\begin{aligned} a - (at + b) &= 2t + 3 \\ -at + (a - b) &= 2t + 3. \end{aligned}$$

Comparing the coefficients, we see that

$$-a = 2 \text{ and } a - b = 3.$$

Hence, $a = -2$ and $b = -5$. Therefore, $y_p = -2t - 5$.

Step3: Write down the final solution.

Using steps 1 and 2, we conclude that the general solution is

$$y = y_p + cy_h = (-2t - 5) + ce^t,$$

where c is an arbritary constant.

Example (Integrating factors)**.** Solve the differential equation

$$ty' + y = e^t$$

Step1: Rewrite the differential equation in the form of $y' + p(t)y = f(t)$.

Dividing the whole equation by t we get that

$$y' + \frac{1}{t}y = \frac{e^t}{t}.$$

Step2: Find $M(t)$ that we want to multiply to the equation.

Once written in the form of $y' + p(t)y = f(t)$, we can compute the function $M(t)$ by the formula

$$M(t) = e^{\int p(t)dt}.$$

In this case we have $p(t) = \frac{1}{t}$ so

$$M(t) = e^{\int \frac{1}{t}dt} = e^{\ln|t|} = t.$$

Step3: Solve for y using integration.

First multiply $M(t) = t$ to the equation to get

$$\begin{aligned} t(y' + \frac{1}{t}y) &= t(\frac{e^t}{t}) \\ ty' + y &= e^t. \end{aligned}$$

But by the choice of $M(t)$, the left hand side will always equal to $\frac{d}{dt}(M(t)y)$. So here we have

$$ty' + y = \frac{d}{dt}(M(t)y) = \frac{d}{dt}(ty).$$

Hence, the equation becomes

$$\frac{d}{dt}(ty) = e^t.$$

Now integrate both sides with respect to t, we get

$$\begin{aligned} \int \frac{d}{dt}(ty)dt &= \int e^t dt \\ ty &= e^t + c \\ y &= \frac{1}{t}(e^t + c). \end{aligned}$$

Example (Growth and decay)**.** A certain material is known to decay at a rate proportional to the amount present. Over a 40-year period, an initial amount of 100 grams has decayed to only 65 grams. Find an expression for the amount of the material t years after the initial measurement.

Step1: Set up the initial value problem.

Let $y(t)$ =the amount of the material t years after the initial measurement (in grams). The statement "it is known to decay at a rate proportional to the amount present" means that

$$\begin{aligned} \text{decay rate} &= k \cdot \text{amount present} \\ \frac{dy}{dt} &= ky \end{aligned}$$

for some decay constant k. Also, the statement "over a 40-year period, an initial amount of 100 grams has decayed to only 65 grams" means that $y(0) = 100$ and $y(40) = 65$. So the setup of this problem is

$$\frac{dy}{dt} = ky, \; y(0) = 100, \; y(40) = 65.$$

Step2: Solve the differential equation and the constants.

We know that the general solution to $\frac{dy}{dt} = ky$ is $y = ce^{kt}$. Using $y(0) = 100$, we get that

$$100 = ce^{k(0)} = c$$

so $y = 100e^{kt}$. To find k we plug in $y(40) = 65$. That is,

$$\begin{aligned} 65 &= 100e^{k(40)} \\ 0.65 &= e^{40k} \\ \ln 0.65 &= 40k \\ k &= \frac{\ln 0.65}{40}. \end{aligned}$$

Therefore, the final solution is

$$y(t) = 100e^{\frac{\ln 0.65}{40}t}.$$

I. Linear differential equations/Linear differential operators

1. Classify the following differential equations. Notice that the last two columns only apply if the differential equation is linear.

	Order?	Linear?	Homogeneous?	Constant/Variable coefficients?
$y'' + e^y = 3t$				
$(\sin t)y^{(5)} + y = 0$				
$4y' - ey = te^t$				
$y^{(3)} + 2y = t + \ln y$				

2. Determine if the following differential operators are linear or non-linear.

a) $L(y) = y' - e^t y$
b) $L(y) = 2y'' + (\tan t)y + 4$
c) $L(y) = y^{(3)} + y'' + y^2 + 1$

II. Undetermined coefficients/Integrating factors

1. Find the general solution to the following differential equations.
a) $y' + y = e^{2t}$
b) $(t^2 + 1)y' + 2ty = 0$
c) $y' - 2y = \frac{1}{1+e^{-2t}}$
d) $y' + y = t^2$

2. Solve the following initial value problems.
a) $y' - y = 1$, $y(0) = 1$
b) $y' + \frac{y}{t-2} = t - 1$, $y(3) = \frac{1}{2}$
7. $y' - 2y = e^{-t}$, $y(0) = 0$

III. Growth and decay

1. (HW5 #5) A certain colony of bacteria grows at a rate proportional to the number of bacteria present. Suppose the number of bacteria doubles every 12 hours. How long (in hours) will it take this colony to grow to five times its original size?

2. (HW5 #8) Upon entering college, Meena borrowed the limit of \$5000 on her credit card to help pay expenses. The credit card company charges 19.95% annual interest, compounded continuously. How much will Meena owe when she graduates in four years?

3. (cf. HW5 #9) Linda has won a lottery consisting of one million dollars. Suppose that she deposits the money in a savings account that pays an annual rate of 8% compounded continuously. How long will the money last if she makes annual withdrawals of $\$100,000$?