MATH 3C LINEAR DIFFERENTIAL EQUATIONS

I. Linear differential equations

Keywords: homogeneous, non-homogeneous, order, constant coefficients, variable coefficients

a) What does it mean for a DE to be linear?

A linear equation in $\mathbb{R}^2$:

A linear equation in space $\mathbb{R}^3$:

A first-order linear differential equation:

A second-order linear differential equation:

An n-th-order linear differential equation:

b) What does it mean for a linear DE to be homogeneous?

Examples.

1. $y' + ty^2 = 1$

Linear or Non-linear   Homogeneous or Non-homogeneous   Order =

2. $e^t y'' + 2y' + y = 0$

Linear or Non-linear   Homogeneous or Non-homogeneous   Order =

3. $y^{(3)} + y'' + y' + y = \sqrt{\sin t}$

Linear or Non-linear   Homogeneous or Non-homogeneous   Order =

4. $ty' + y = \sin y$

Linear or Non-linear   Homogeneous or Non-homogeneous   Order =

II. Solutions to a linear differential equation

Goal: Suppose we can find some particular solutions to a linear differential equation. We want to come up with a way to create many more other solutions based on the particular solutions.

Main idea: We want to make use of the linearity of the differential equation and the linearity of derivatives. (There are two linear properties of derivatives. What are they?)

a) Homogeneous

Consider the second-order homogeneous liner DE

$$y'' + 3y' + 2y = 0.$$

Suppose I tell you that the functions $y = e^{-2t}$ and $y = e^{-t}$ are two particular solutions.

1. How would you find more solutions to this linear DE using these two functions?

2. Now think of the general case. If $y = f_1(t)$ and $y = f_2(t)$ are two particular solutions to the homogeneous linear DE

$$a_n(t)y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0,$$

how would you create more solutions based on them?

b) Non-homogeneous

Consider the first-order non-homogeneous linear DE

$$y' + 2y = 3e^t.$$

Suppose I tell you that $y = e^t$ is a particular solution to the above DE, and that the function $y = e^{-2t}$ is a particular solution to the corresponding homogeneous linear DE

$$y' + 2y = 0.$$

1. How would you find more solutions to the non-homogeneous DE using these two functions?

2. Now think of the general case. If $y = f_p(t)$ is a particular solution to the non-homogeneous linear DE

$$a_n(t)y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = g(t)$$

and $y = f_h(t)$ is a particular solution to the corresponding homogeneous linear DE

$$a_n(t)y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0,$$

how would you create more solutions from them?

III. Finding a particular solution

Let's look at a very special case. We will consider a **second-order homogeneous linear DE with constant coefficients**. How can we solve for a particular solution?

Consider the DE

$$y'' + 5y' + 6y = 0.$$

1. Find two particular solutions to this DE. (Hint: Start with $y = e^{rt}$. What can r be?)

2. Write down a family of solutions using the two solutions you found in part 1.