## Math 3C Inverse, Determinant, and Cramer's Rule

### I. Inverse of a square matrix

a) Inverse of a $2 \times 2$ matrix.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} =$$

b) Inverse of a $3 \times 3$ matrix.

$$\begin{bmatrix} 3 & 3 & -6 \\ 0 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}^{-1} =$$

c) Solving a linear system.

$$\begin{aligned} 3x + 3y - 6z &= 1 \\ y + z &= -1 \\ -2y + z &= 0 \end{aligned}$$

**Check your understanding.**

Let $\mathbf{A}$ be an $n \times n$ matrix.

| | $\mathbf{A}$ is invertible | $\mathbf{A}$ is not invertible |
|---|---|---|
| $\mathbf{A}$ is row equivalent to $I_n$ | true false | true false |
| $\mathbf{A}^{\mathbf{T}}$ is invertible | true false | true false |
| the rank $r$ of $\mathbf{A}$ | $r < n$ $r = n$ | $r < n$ $r = n$ |
| solutions to $\mathbf{Ax} = \mathbf{0}$ | none unique infinitely many | none unique infinitely many |
| solutions to $\mathbf{Ax} = \mathbf{b}$ | none unique infinitely many | none unique infinitely many |

## II. Determinant

a) Determinant of a $2 \times 2$ matrix.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} =$$

b) Row operations and Determinants.

i. $R_i \leftrightarrow R_j$ (interchanging two rows)

ii. $R_i \to kR_i$ (multiplying a row by a constant $k$)

iii. $R_i \to R_i + kR_j$ (adding a multiple of a row to another)

c) Determinant of a $3 \times 3$ matrix.

$$\begin{vmatrix} 3 & 3 & -6 \\ 0 & 1 & 1 \\ 0 & -2 & 1 \end{vmatrix} =$$

d) Invertibility and Determinant.

e) A very important property of determinant.

**Check your understanding**.

1. Let $\mathbf{A}$ be an $n \times n$ matrix. Determine if the following statements are true or false.
a) If $\mathbf{A}$ has a row of zero, then $|\mathbf{A}| = 0$.
b) If $\mathbf{A}$ has two identical rows, then $\mathbf{A}$ is not invertible.
c) If $|\mathbf{A}| \neq 0$ then $\mathbf{Ax} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbb{R}^n$.

2. Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ matrices. Use the property in e) to prove that

a) $|\mathbf{AB}| = |\mathbf{BA}|$

b) $|\mathbf{A^{-1}}| = \frac{1}{|\mathbf{A}|}$ (provided that $\mathbf{A}$ is invertible)

c) $|\mathbf{BAB^{-1}}| = |\mathbf{A}|$ (provided that $\mathbf{B}$ is invertible)

### III. Cramer's rule

**Statement.** Let $\mathbf{A}$ be an $n \times n$ matrix with $|\mathbf{A}| \neq 0$. Then $\mathbf{A}$ is an invertible matrix so the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$ in $\mathbb{R}^n$. Denote by $\mathbf{A_i}$ the matrix obtained from $\mathbf{A}$ by replacing its $i$th column with the column vector $\mathbf{b}$. Cramer's rule says that the $i$th component of the unique solution of the system $\mathbf{Ax} = \mathbf{b}$ is given by $x_i = |\mathbf{A_i}|/|\mathbf{A}|$.

**Example.** This is the system we considered above in part I.

$$\begin{aligned} 3x + 3y - 6z &= 1 \\ y + z &= -1 \\ -2y + z &= 0. \end{aligned}$$

In matrix form, this is represented by

$$\mathbf{Ax} = \begin{bmatrix} 3 & 3 & -6 \\ 0 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \mathbf{b}.$$

We saw in part II that $|\mathbf{A}| = 9$. Now, replacing the $i$th column with $\mathbf{b}$, we have

$$\mathbf{A_1} = \begin{bmatrix} 1 & 3 & -6 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}, \ \mathbf{A_2} = \begin{bmatrix} 3 & 1 & -6 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \ \mathbf{A_3} = \begin{bmatrix} 3 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & -2 & 0 \end{bmatrix}.$$

You should check that $|\mathbf{A_1}| = -6$, $|\mathbf{A_2}| = -3$, and $|\mathbf{A_3}| = -6$. By Cramer's rule, the solution is

$$x = \frac{|\mathbf{A_1}|}{|\mathbf{A}|} = \frac{-6}{9} = \frac{-2}{3}, \ y = \frac{|\mathbf{A_2}|}{|\mathbf{A}|} = \frac{-3}{9} = \frac{-1}{3}, \ z = \frac{|\mathbf{A_3}|}{|\mathbf{A}|} = \frac{-6}{9} = \frac{-2}{3}.$$

Hence, $(x, y, z) = (\frac{-2}{3}, \frac{-1}{3}, \frac{-2}{3})$, which agrees with what we got earlier by using the inverse of $\mathbf{A}$.