MATH 4A FALL 2015 WORKSHEET 5

Name: Section: 8AM 5PM 6PM 7PM

*The first three problems are worth 10 points each. From Problem 4 and on, each answer is worth 1 point. You need 35 points to get 1% on your discussion section grade.

1. Complete the definition: A set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of vectors in $\mathbb{R}^n$ is said be *linearly independent* if

2. Complete the definition: A set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of vectors in $\mathbb{R}^n$ is said be *linearly dependent* if

3. Complete the definition: A *linear transformation* $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is

The *domain of* T is and the *codomain of* T is .

Every matrix gives rise to a linear transformation $\mathbb{R}^n \longrightarrow \mathbb{R}^m$.

Example. The matrix A below defines a linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

In order for $A\mathbf{x}$ to make sense, the vector $\mathbf{x}$ must have three entries. This means that $\boxed{\text{the domain of } T \text{ is } \mathbb{R}^3}$. Moreover, given a vector $\mathbf{x} \in \mathbb{R}^3$, we have

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

So $T(\mathbf{x})$ is just a linear combination of the columns of A. This means that $\boxed{\text{the codomain of } T \text{ is } \mathbb{R}^2}$.

4. Let T be the linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by the matrix A below.

$$A = \begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 5 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$$

a) What is n? b) What is m?

c) Compute the following or indicate that it does not make sense.

(i) $T(\mathbf{u})$ (ii) $T(\mathbf{v})$

Every linear transformation $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ is given by a matrix.

Example. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the linear transformation defined by $T(x_1, x_2) = (x_1 + x_2, 3x_1 - x_2, 4x_1)$. Note that we can rewrite this as

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 3x_1 - x_2 \\ 4x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This means that T is represented by the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \\ 4 & 0 \end{bmatrix}.$$

Also, observe that the first column of A is $T(1, 0)$ and the second column of A is $T(0, 1)$. In general, $\boxed{\text{the columns of the matrix } A \text{ associated to a linear transformation } T \text{ is just the images the standard basis vectors under } T}$.

5. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation defined by $T(x_1, x_2, x_3) = (x_1 - 2x_2, x_2 + x_3, x_1 - 3x_3)$. Find the matrix associated to T.

6. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear transformation such that $T(2, 0) = (0, 2, 4)$ and $T(0, 1) = (-1, 4, 5)$.

a) Find the matrix associated to T.

b) Find $T(3, 2)$.

The concept of "onto" and "one-to-one" Consider a map (not necessarily linear) $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

- T is *onto* if the equation $T(\mathbf{x}) = \mathbf{b}$ has $\boxed{\text{at least one solution}}$ for $\boxed{\text{every vector } \mathbf{b}}$ in $\mathbb{R}^m$.
- T is *one-to-one* if the equation $T(\mathbf{x}) = \mathbf{b}$ has $\boxed{\text{at most one solution}}$ for $\boxed{\text{every vector } \mathbf{b}}$ in $\mathbb{R}^m$.

Example. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ be a linear transformation whose matrix A is equivalent to the matrix

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

which is in echelon form. We see that for some vectors $\mathbf{b} \in \mathbb{R}^4$, the augmented matrix $[A \mid b]$ row reduces to a matrix containing a row that looks like $[\,0\ \ 0\ \ 0 \mid *\,]$ with $* \neq 0$. This means that $A\mathbf{x} = \mathbf{b}$ has no solution for these $\mathbf{b}$ and so $\boxed{A \text{ is not onto}}$. On the other hand, for all vectors $\mathbf{b} \in \mathbb{R}^4$, the augmented matrix $[A \mid b]$ has no free variable. This means that $A\mathbf{x} = \mathbf{b}$ has at most one solution for any $\mathbf{b}$ and so $\boxed{A \text{ is one-to-one}}$.

7. Determine whether the linear transformation T in Problem 5 is (i) onto and (ii) one-to-one. Show your work.