Homework 10 Selected Solutions

1k) Let $f : [2, 3) \to [0, \infty)$ be given by $f(x) = \frac{x-2}{3-x}$. Is f onto?

Scratch Work. Given $y \in [0, \infty)$, we need to find $x \in [2, 3)$ with $f(x) = y$.

$$\begin{aligned} y &= \frac{x-2}{3-x} \\ 3y - xy &= x - 2 \\ 3y + 2 &= x + xy \\ x &= \frac{3y+2}{1+y}. \end{aligned}$$

You should not include the above in your proof because you can't start your proof with the conclusion.

Solution. The function f is onto. To see why, let $y \in [0, \infty)$. We take

$$x = \frac{3y+2}{1+y},$$

which is well-defined because $y \neq -1$. First we check that $x \in [2, 3)$. Indeed,

$$x = \frac{2(y+1)+y}{1+y} = 2 + \frac{y}{1+y},$$

and since $y \geq 0$, we have

$$0 \leq \frac{y}{1+y} < 1.$$

This shows that $2 \leq x < 3$, as desired. Next we check that $f(x) = y$. Indeed,

$$\begin{aligned} x &= \frac{3y+2}{1+y} \\ x + xy &= 3y + 2 \\ 3y - xy &= x - 2 \\ y &= \frac{x-2}{3-x}, \end{aligned}$$

where we can divide by $3 - x$ because $x \neq 3$. This proves that f is onto.

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5) Prove that if $f : A \to B$ nd $g : B \to C$ are onto, then so is $g \circ f : A \to C$.

Solution. Let $c \in C$ be given. Since g is surjective, we have $g(b) = c$ for some $b \in B$. Now, since f is surjective, we have $f(a) = b$ for some $a \in A$. But then $(g \circ f)(a) = g(f(a)) = g(b) = c$, which shows that $g \circ f$ is surjective.

10b) Prove that $f : \mathbb{R} \to \mathbb{R}$ defined as below is onto but not injective.

$$f(x) = \begin{cases} x + 4 & \text{if } x \leq -2 \\ -x & \text{if } -2 < x < 2 \\ x - 4 & \text{if } x \geq 2. \end{cases}$$

Scratch Work. First you should graph the function. Then, from the graph, you will see that the $x - 4$ portion of the graph will give you all the y-values greater than or equal to -2, in which case $y = x - 4$ so $x = y + 4$ would work. For the y-values which are less than -2, from the graph they are covered by the $x + 4$ portion, in which case $y = x + 4$ so $x = y - 4$ works.

Solution (for surjectivity). Let $y \in \mathbb{R}$.

Case (i). $y \geq -2$: Take $x = y + 4$, and observe that $x \geq -2 + 4 = 2$. So

$$f(x) = x - 4 = (y + 4) - 4 = y.$$

Case (ii). $y < -2$: Take $x = y - 4$, and observe that $x < -2 - 4 = -6$. So

$$f(x) = x + 4 = (y - 4) + 4 = y.$$

In either case, we have $f(x) = y$ for some $x \in \mathbb{R}$, whence f is onto.

**Note: We could also have considered the cases $y \leq 2$ and $y > 2$ (how?).

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12) **Hint:** Use the Pigeon hole principle to show non-surjectivity in a) and non-injectivity in b). For c), because the domain and codomain have the same (finite) number of elements, if you can show that it is one-to-one, then it is onto automatically, and vice versa.
