7b) Suppose $a$ and $b$ be positive integers. Prove that

\[ a + 1 \text{ divides } b \text{ and } b \text{ divides } b + 3 \text{ if and only if } a = 2 \text{ and } b = 3. \]

**Solution.** First assume that $a + 1$ divides $b$ and $b$ divides $b + 3$. Then,

\[ b = m(a + 1) \quad \text{and} \quad b + 3 = bn \]

for some $m, n \in \mathbb{Z}$. From $b + 3 = bn$, we deduce that

\[ 3 = b(n - 1) \]

and so $b$ divides 3. Since $b > 0$, we have $b = 1$ or $b = 3$. If $b = 1$, then

\[ 1 = b = m(a + 1) \]

and so $a + 1$ divides 1. Hence, $a + 1 = \pm 1$, which is impossible because $a > 0$ implies that $a + 1 > 1$. This shows that $b = 1$ is not possible and so we must have $b = 3$. In this case, we have

\[ 3 = b = m(a + 1) \]

and so $a + 1$ divides 3. Since $a + 1 > 1$, we have $a + 1 = 3$ and so $a = 2$. This proves that $b = 3$ and $a = 2$.

Conversely, assume that $a = 2$ and $b = 3$. Then, $a + 1 = 3$, which divides

\[ b = 3, \quad \text{and} \quad b + 3 = 6 \]

which is divisible by $b = 3$.

11) Three numbers, $x$, $y$, and $z$, are chosen between 0 and 1. Suppose that $0 < x < y < z < 1$. Prove that at least two of the numbers $x$, $y$, and $z$ are within 1/2 unit from one another.

**Solution.** Suppose that $x, y, z \in \mathbb{R}$ are such that $0 < x < y < z < 1$. Suppose on the contrary that no two of the numbers $x$, $y$, and $z$ are within 1/2 unit from one another, i.e.

\[ y - x \geq 1/2, \quad z - y \geq 1/2, \quad z - x \geq 1/2. \]
We then deduce that

\[ 1 = (1 - z) + (z - y) + (y - x) + (x - 0) \]
\[ \leq (1 - z) + 1/2 + 1/2 + x \]
\[ = 1 + (1 - z) + x. \]

Since \(0 < x < z < 1\) by hypothesis, we have \(x > 0\) and \(1 - z > 0\). Thus,

\[ 1 \leq 1 + (1 - z) + z < 1, \]

which implies that \(1 < 1\). This is a contradiction. Thus, at least two of the numbers \(x, y,\) and \(z\) are within \(1/2\) unit from one another.