Homework 4 Selected Solutions

7b) Suppose a and b be positive integers. Prove that

$a+1$ divides b and b divides $b+3$ if and only if $a=2$ and $b=3$.

Solution. First assume that $a+1$ divides b and b divides $b+3$. Then,

$$b = m(a+1) \qquad \text{and} \qquad b+3 = bn$$

for some $m, n \in \mathbb{Z}$. From $b+3 = bn$, we deduce that

$$3 = b(n-1)$$

and so b divides 3. Since $b > 0$, we have $b = 1$ or $b = 3$. If $b = 1$, then

$$1 = b = m(a+1)$$

and so $a+1$ divides 1. Hence, $a+1 = \pm 1$, which is impossible because $a > 0$ implies that $a+1 > 1$. This shows that $b = 1$ is not possible and so we must have $b = 3$. In this case, we have

$$3 = b = m(a+1)$$

and so $a+1$ divides 3. Since $a+1 > 1$, we have $a+1 = 3$ and so $a = 2$. This proves that $b = 3$ and $a = 2$.

Conversely, assume that $a = 2$ and $b = 3$. Then, $a+1 = 3$, which divides $b = 3$, and $b+3 = 6$, which is divisible by $b = 3$.

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11) Three numbers, x, y, and z, are chosen between 0 and 1. Suppose that $0 < x < y < z < 1$. Prove that at least two of the numbers x, y, and z are within $1/2$ unit from one another.

Solution. Suppose that $x, y, z \in \mathbb{R}$ are such that $0 < x < y < z < 1$. Suppose on the contrary that no two of the numbers x, y, and z are within $1/2$ unit from one another, i.e.

$$y - x \geq 1/2, \quad z - y \geq 1/2, \quad z - x \geq 1/2.$$

We then deduce that

$$\begin{aligned} 1 &= (1-z)+(z-y)+(y-x)+(x-0) \\ &\leq (1-z)+1/2+1/2+x \\ &= 1+(1-z)+x. \end{aligned}$$

Since $0 < x < z < 1$ by hypothesis, we have $x > 0$ and $1-z > 0$. Thus,

$$1 \leq 1+(1-z)+z < 1,$$

which implies that $1 < 1$. This is a contradiction. Thus, at least two of the numbers x, y, and z are within $1/2$ unit from one another.
