Homework 7 Selected Solutions

3f) $\text{Dom}(W) = (-2, 2) \qquad \text{Rng}(W) = \{3\}$
3g) $\text{Dom}(W) = \mathbb{R} \qquad \text{Rng}(W) = \mathbb{R}$
6d) $R_4^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \sqrt{x-2} \text{ or } y = -\sqrt{x-2}\}$
7f) $T \circ T = \{(1, 1), (4, 4)\}$

Recall the definitions of domain, range, and composition:
Let R be a relation from A to B.

$$\text{Dom}(R) = \{x \in A \mid (x, y) \in R \text{ for some } y \in B\}$$

$$\text{Rng}(R) = \{y \in B \mid (x, y) \in R \text{ for some } x \in A\}$$

Further let S be a relation from B to C.

$$S \circ R = \{(x, z) \in A \times C \mid (x, y) \in R \text{ and } (y, z) \in S \text{ for some } y \in B\}$$

Observations.
(i) $\text{Dom}(R) \subset A$ and $\text{Rng}(R) \subset B$ directly from the definitions.
(ii) Given $x \in A$, from the definition of domain, we have

$$x \in A \iff (x, y) \in R \text{ for some } y \in B.$$

(iii) Given $y \in B$, from the definition of range, we have

$$y \in A \iff (x, y) \in R \text{ for some } x \in A.$$

(iv) Given $(x, z) \in A \times B$, from the definition of composition, we have

$$(x, z) \in S \circ R \iff (x, y) \in R \text{ and } (y, z) \in S \text{ for some } y \in B.$$

5c) Let $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : 9x^2 + 4y^2 < 36\}$. Show $\text{Rng}(R) = (-3, 3)$.
Solution. There are two inclusions to prove.
(1) $(-3, 3) \subset \text{Rng}(R)$: Let $y \in (-3, 3)$. Then, choosing $x = 0$, we have

$$9x^2 + 4y^2 = 4y^2 < 36 \qquad (\text{since } y^2 < 9).$$

This shows that $(0, y) \in R$ and so $y \in \text{Rng}(R)$.

(2) $\text{Rng}(R) \subset (-3,3)$: Let $y \in \text{Rng}(R)$. By definition, there exists $x \in \mathbb{R}$ such that $(x,y) \in R$, i.e. $9x^2 + 4y^2 < 36$. Sinec $x^2 \geq 0$, we deduce that

$$4y^2 \leq 9x^2 + 4y^2 < 36,$$

Hence, $y^2 < 9$ and so $y \in (-3,3)$.

Combining (1) and (2), we see that $\text{Rng}(R) = (-3,3)$.

In (2) of the proof, the x is given to you so you cannot assume $x = 0$.

5d) Let $R = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x = |y|\}$. Show that

$$(i) \quad \text{Dom}(R) = [0,\infty) \qquad \text{and} \qquad (ii) \quad \text{Rng}(R) = \mathbb{R}.$$

Solution.

(i) There are two inclusions to prove.

(1) $[0,\infty) \subset \text{Dom}(R)$: Let $x \in [0,\infty)$. Then, choosing $y = x$, we have

$$|y| = |x| = x \qquad (\text{since } x \geq 0).$$

This show that $(x,x) \in R$ and so $x \in \text{Dom}(R)$.

(2) $\text{Dom}(R) \subset [0,\infty)$: Let $x \in \text{Dom}(R)$. By definition, there exists $y \in \mathbb{R}$ such that $(x,y) \in R$, i.e. $x = |y|$. By definition of absolute value, $|y| \geq 0$. Thus, $x \geq 0$ and so $x \in [0,\infty)$.

Combining (1) and (2), we see that $\text{Dom}(R) = [0,\infty)$.

(ii) Since $\text{Rng}(R) \subset \mathbb{R}$ by definition of the range (see Observation (i) above), we only have to prove that $\mathbb{R} \subset \text{Rng}(R)$. So let $y \in \mathbb{R}$. Choosing $x = |y|$, we have that $(x,y) \in R$. This shows that $y \in \text{Rng}(R)$. Thus, $\text{Rng}(R) = \mathbb{R}$.

11b) Let R be a relation from A to B and S a relation from B to C. Prove that $\text{Dom}(S \circ R) \subset \text{Dom}(R)$. (The proof of $\text{Rng}(S \circ R) \subset \text{Rng}(S)$ is similar.)

Solution. Let $x \in \text{Dom}(S \circ R)$. By definition, we have

$$(x,z) \in S \circ R \qquad \text{for some } z \in C.$$

Now, by definition of $S \circ R$, this implies that

$$(x,y) \in R \text{ and } (y,z) \in S \text{ for some } y \in B.$$

Since $(x,y) \in R$, we see that $x \in \text{Dom}(R)$. Thus, $\text{Dom}(S \circ R) \subset \text{Dom}(R)$.
