12.4. Source Functions

Fourier Transform can be used in finding the source function of a PDE from scratch.

Diffusion equation:
The source function is defined as the unique solution of
\[
\begin{align*}
S_t &= S_{xx}, \\
S(x,0) &= S(x)
\end{align*}
\]
where the diffusion constant is $1$. Let's assume we know nothing about the form of $S(x,t)$. We only assume it has a Fourier transform as a distribution in $x$, for each $t$:
\[
\hat{S}(k,t) = \int_{-\infty}^{\infty} S(x,t) e^{-ikx} \, dx.
\]
Here $k$ denotes the frequency variable.

By property (i) of Fourier transforms, the PDE takes the form
\[
\frac{\partial \hat{S}}{\partial t} = (ik)^2 \hat{S} = -k^2 \hat{S}
\]
\[
\hat{S}(k,0) = 1
\]

For each $k$, this is an ODE with the solution \( \hat{S}(k, t) = e^{-k^2t} \).

Then
\[
S(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-k^2t} e^{ikx} \, dk = \frac{1}{\sqrt{4\pi t}} \, e^{-\frac{x^2}{4t}}
\]
which agrees with §2.4.

Wave equation:
The source function for the 1D wave equation satisfies
\[
\begin{align*}
S_{tt} &= c^2 S_{xx} \\
S(x,0) &= 0 \\
S_t(x,0) &= S(x)
\end{align*}
\]

Using Fourier Transform, we get
\[
\begin{align*}
\frac{\partial^2 \hat{S}(k, t)}{\partial t^2} &= -c^2 k^2 \hat{S}(k, t) \\
\hat{S}(k,0) &= 0 \\
\frac{\partial \hat{S}(k,0)}{\partial t} &= 1
\end{align*}
\]
\[
\hat{S}(k, t) = \frac{1}{k^2} \sin(kc) \Rightarrow S(x,t) = \frac{1}{k} \sin(kc) t
\]
\[ \hat{S}(k,t) = \frac{e^{i k c t} - e^{-i k c t}}{2i k c} \]

\[ S(x,t) = \frac{\frac{\text{sgn}(x+ct)-\text{sgn}(x-ct)}{4c}}{i k c} \]

According to the table, the transform of \( \text{sgn}(x) = H(x) - H(-x) \) is \( \frac{2}{ik} \). By property (iii), the transform of \( \frac{e^{-i k c t}}{2i k c} \) is \( \frac{1}{2i k c} \).

\[ S(x,t) = \frac{\frac{\text{sgn}(x+ct)-\text{sgn}(x-ct)}{4c}}{i k c} = \begin{cases} 0 & \text{for } \ |x| > ct \\ \frac{1}{2c} & \text{for } \ |x| < ct \end{cases} \]

In 3D, the source function has a Fourier Transform

\[ S^3(k,t) \]

\[ \begin{align*}
\frac{\partial^2}{\partial t^2} S &= -c^2 (k_x^2 + k_y^2 + k_z^2) S \\
S(k,0) &= 0 \\
\frac{\partial S}{\partial t}(k,0) &= 1
\end{align*} \]

where \( k = (k_x, k_y, k_z) \) and \( k^2 = (k_x^2 + k_y^2 + k_z^2) \).

\[ S(x,t) = \iiint_{R^3} \frac{1}{k c} \sin k c t \ e^{i k \cdot x} \ rac{dk}{(2\pi)^3} \]

Spherical coordinates:

\[ S(x,t) = \iiint_0^{2\pi} \int_0^1 \int_0^{\infty} \frac{1}{k c} \sin k c t \ e^{i k r \cos \theta} k^2 \sin \theta \ \frac{dk \, d\theta \, d\phi}{8\pi^3} \]

Notice how the characteristic variables show up again!

For \( t > 0 \), \( ct + r > 0 \)

\[ S(x,t) = \frac{1}{4\pi c t} \frac{1}{4\pi c t} S(\sqrt{(ct - r)^2 + x^2}) \]

which agrees with our previous answer for \( t > 0 \).
Laplace’s equation in a half-plane:
\[
\begin{align*}
\begin{cases}
 u_{xx} + u_{yy} &= 0 \quad y > 0 \\
 u(x, 0) &= f(x) \quad y = 0
\end{cases}
\end{align*}
\]

We cannot transform variable since \( y > 0 \).

Define \( U(k, y) = \int_{-\infty}^{\infty} u(x, y) e^{-ikx} \, dx \)

then \( U \) satisfies the ODE:
\[
\begin{align*}
\begin{cases}
 -k^2 U + U_{yy} &= 0 \quad y > 0 \\
 U(k, 0) &= 1
\end{cases}
\end{align*}
\]

The solutions are \( e^{\pm iky} \). We must reject solutions growing exponentially as \( |k| \to \infty \) so \( U(k, y) = e^{-iky} \).

\[
U(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iky} e^{ikx} \, dk
\]

\[
= \frac{1}{2\pi} \left( \int_{-\infty}^{0} e^{-iky} e^{ikx} \, dk + \int_{0}^{\infty} e^{-iky} e^{ikx} \, dk \right)
\]

\[
= \frac{1}{2\pi} \left( \frac{e^{iky}}{ik} \right) \bigg|_{-\infty}^{0} + \frac{1}{2\pi} \left( \frac{e^{-iky}}{-ik} \right) \bigg|_{0}^{\infty}
\]

\[
= \frac{1}{2\pi} \left( \frac{1}{y - ix} + \frac{1}{y + ix} \right) = \frac{y}{\pi(x^2 + y^2)}
\]

Green’s Function for the half-plane:
\[
G(x, y; x_0, y_0) = \frac{1}{2\pi} \log \left| \frac{y - y_0}{y - y_0} \right| - \frac{1}{2\pi} \log \left| x - x_0 \right|
\]

satisfies (i) (ii) (iii) properties.

\[
G(x, y; x_0, y_0) = \frac{1}{4\pi} \left( \log \left| (x - x_0) + (y - y_0)^2 \right| - \log \left| (x - x_0) + (y - y_0)^2 \right| \right)
\]

\[
\frac{\partial G}{\partial n} = -\frac{y_0}{\pi(x^2 + y_0^2)}
\]

when \( y = 0 \).

\[
U(x_0, y_0) = \int_{-\infty}^{\infty} u(x, 0) \frac{\partial G}{\partial n} \bigg|_{y = 0} \, dx = \int_{-\infty}^{\infty} S(x) \frac{\partial G}{\partial n} \bigg|_{y = 0} \, dx
\]

\[
= \frac{y_0}{\pi(x_0^2 + y_0^2)}
\]

which coincides with the solution obtained by Fourier transform.