# Math 108B Selected Homework Solutions

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## Homework 1

5.1.7 (a) If matrices $A, B$ both represent $T$ under different bases, then for some invertible matrix $Q$ we have $B = QAQ^{-1}$. Then

$$\det(B) = \det(QAQ^{-1}) = \det(Q) \cdot \det(A) \cdot \det(Q^{-1}),$$

since the determinant for matrices is multiplicative. Furthermore $\det(Q^{-1}) = (\det(Q))^{-1}$, so

$$\det(B) = \det(A),$$

as desired.

(b) We know that $T$ is invertible if and only if its matrix is invertible with respect to any basis; we also know that a matrix is invertible if and only if its determinant is zero. If $A$ is a matrix representing $T$ in some basis, then $T$ is invertible if and only if $A$ is invertible if and only if $\det(T) = \det(A) \neq 0$.

(c) If $T$ has matrix $A$ under some basis, then $T^{-1}$ has matrix $A^{-1}$ under the same basis. Thus

$$\det(T^{-1}) = \det(A^{-1}) = \det(A)^{-1} = \det(T)^{-1}.$$

(d) If operators $T, U$ are represented by matrices $A, B$ (respectively) in some basis, then the matrix for $TU$ is $AB$. From this we have

$$\det(TU) = \det(AB) = \det(A)\det(B) = \det(T)\det(U),$$

since the determinant for matrices is multiplicative.

(e) Given any ordered basis $\beta = \{v_1, \ldots, v_n\}$, the identity map $I_V$ maps each $v_i$ to itself. That is, $[I_V]_\beta = I$. Matrix subtraction is defined so that $[T - \lambda I_V]_\beta = [T]_\beta - \lambda [I_V]_\beta = [T]_\beta - \lambda I$. Therefore

$$\det(T - \lambda I_V) = \det([T - \lambda I_V]_\beta) = \det([T]_\beta - \lambda I).$$

## Homework 2

5.1.8 Let $T : V \to V$ be a linear operator.

(a) ($\Rightarrow$) Assume that $T$ is invertible and $Tv = 0$ for some vector $v$. Multiplying by $T^{-1}$ gives $v = 0$, so the equation $Tv = 0v$ has only trivial solutions; that is, zero is not an eigenvalue.

($\Leftarrow$) Suppose that the equation $Tv = 0$ has only the trivial solution $v = 0$. That is, $\ker(T) = \{0\}$. Since $V$ is finite–dimensional, this implies $T$ is invertible.

(b) Suppose that $T$ is invertible and that $Tv = \lambda v$ for some $\lambda \neq 0$ and $v \neq 0$. Multiplying by $T^{-1}$ and $\lambda^{-1}$ gives $T^{-1}v = \lambda^{-1}v$. Multiplying by $T$ and $\lambda$ reverses the argument.

5.1.14 We know that $\det(A) = \det(A^t)$ for all matrices $A$; furthermore $(\lambda I)^t = \lambda I$. Since the transpose operation is additive we have

$$\det(A - \lambda I) = \det((A - \lambda I)^t) = \det(A^t - \lambda I).$$

5.1.15a If $Tv = \lambda v$ for some nonzero $v$, then $T^2 v = T(Tv) = \lambda(Tv) = \lambda^2 v$. Similarly, $T^n v = \lambda T^{n-1} v$ for each $n$; an inductive argument shows that $T^n v = \lambda^n v$ for all integers $n \geq 0$.

5.2.1 (a) False. For example, the identity map on $\mathbb{R}^3$ has only the eigenvalue 1.

(b) False. For example, for the identity map on $\mathbb{R}^2$ the standard basis vectors $e_1$ and $e_2$ are eigenvectors with eigenvalue 1, though $\{e_1, e_2\}$ is linearly independent.

(c) False. We never consider 0 to be an eigenvector, though it must be in any eigenspace.

(d) True. If $v$ satifies $Tv = \lambda_i v$ for $i = 1, 2$, then $(\lambda_1 - \lambda_2)v = 0$; if $\lambda_1 \neq \lambda_2$, then $v = 0$.

(e) True.

(f) False. The eigenvalues might not exist in the field. For example, the 90° rotation of $\mathbb{R}^2$ has no eigenvalues in $\mathbb{R}$. It is not diagonalizable over $\mathbb{R}$, even though "every eigenvalue $\lambda$ has multiplicity equal to $\dim(E_\lambda)$," a statement which holds vacuously.

(g) True.

(h) True.

(i) False. For example take three distinct lines through the origin in $\mathbb{R}^2$. Each pair has a trivial intersection, but the sum of any two is the entire plane (which intersects the third line nontrivially).

5.2.20 Throughout the following, $V$ is a finite–dimensional vector space with subspaces $W_1, \ldots, W_k$ satisfying

$$V = \sum_{i=1}^{k} W_i.$$

Note that with no further assumptions we have

$$\dim(V) \leq \sum_{i=1}^{k} \dim(W_i),$$

which can be proved with induction and the readily–verified result

$$\dim(W_1 + W_2) \leq \dim(W_1) + \dim(W_2).$$

Let's give a brief proof of the above inequality. Given bases $\beta = \{v_1, \ldots, v_m\}$ and $\gamma = \{w_1, \ldots, w_n\}$ for $W_1$ and $W_2$ respectively, we claim that $\beta \cup \gamma$ spans $W_1 + W_2$. Indeed, given an arbitrary vector $x + y \in W_1 + W_2$ with $x \in W_1$ and $y \in W_2$, we find scalars so that $x = \sum c_i v_i$ and $y = \sum d_i w_i$. Then

$$x + y = (c_1 v_1 + \cdots + c_m v_m) + (d_1 w_1 + \cdots + d_n w_n) \in \operatorname{span}(\beta \cup \gamma).$$

Thus the size of a basis for $W_1 + W_2$ must be at most the size of $\beta \cup \gamma$, and we have

$$\dim(W_1 + W_2) \leq |\beta \cup \gamma| \leq |\beta| + |\gamma| = \dim(W_1) + \dim(W_2).$$

($\Rightarrow$) Assume that $V = W_1 \oplus \cdots \oplus W_k$. By theorem 5.10d, if we take an ordered basis $\beta_i$ for each space $W_i$, then $\beta = \beta_1 \cup \cdots \cup \beta_k$ is a basis for $V$. This gives

$$|\beta| = \dim(V) \leq \sum_{i=1}^{k} \dim(W_i) = \sum_{i=1}^{k} |\beta_i|.$$

Thus it suffices to prove that $|\beta| = \sum |\beta_i|$. Suppose that $v \in \beta_1 \cap \beta_2$ and note that we can write the zero vector nontrivially as

$$0 = v + (-v) + 0 + \cdots + 0 \in W_1 + W_2 + \cdots + W_k.$$

This contradicts our assumption that the sum $\sum W_i$ is direct. Hence $\beta_1 \cap \beta_2 = \varnothing$; the same argument shows that $\beta_i \cap \beta_j = \varnothing$ for any $i \neq j$. Since the sets are pairwise disjoint, we have $|\beta| = \sum |\beta_i|$, as desired.

($\Leftarrow$) Now we assume that

$$\dim(V) = \sum_{i=1}^{k} \dim(W_i).$$

For each $W_i$ find an ordered basis $\beta_i$ and consider the set $\beta = \beta_1 \cup \cdots \cup \beta_k$; we do not know a priori that the bases $\beta_i$ are pairwise disjoint, so we only have an inequality:

$$|\beta| \leq \sum_{i=1}^{k} |\beta_i| = \sum_{i=1}^{k} \dim(W_i). \tag{1}$$

We claim that $\beta$ spans $V$. Indeed, given $x \in V = \sum W_i$ we can write

$$x = w_1 + \cdots + w_k$$

with $w_i \in W_i$ for each $i$. Each $w_i$ can be written as a linear combination of the elements of $\beta_i$, so $x$ can be written as a linear combination of the elements of $\beta$ — we could use explicit notation for this argument, but it would greatly clutter the simple idea at work. Since $\beta$ spans $V$, it must have size at least $\dim V$; combining this with equation (1) gives

$$\dim(V) \leq |\beta| \leq \sum_{i=1}^{k} \dim(W_i) = \dim(V).$$

We deduce that $|\beta| = \dim(V)$, and since $\beta$ spans $V$ we further conclude that $\beta$ is a basis of $V$. By theorem 5.10d, $V$ is the direct sum of the spaces $W_i$.

## Homework 3

5.4.1 (a) False. Any operator $T : V \to V$ has $T(V) \subseteq V$.

(b) True. The roots of the characteristic polynomial are eigenvalues. An eigenvalue for $T_W$ is an eigenvalue for $T$, so the roots of the characteristic polynomial of $T_W$ are among those of the polynomial for $T$.

(c) False. We could have, for example, $v' = 2v$.

(d) False. The $T$–cyclic subspace generated by $v$ might not include $v$ in its span. For example, take $T = 0$ on $\mathbb{R}^2$ and any vector $v \neq 0$. The $T$–cyclic subspace generated by $v$ is the span of $v$, but the $T$–cyclic subspace of $Tv$ is trivial.

(e) True. This follows from the Cayley–Hamilton theorem.

(f) True. The matrix on displayed on page 316 has characteristic polynomial $(-1)^n(a_0 + a_1 t + \cdots + a_{k-1}t^{k-1} + t^k)$.

(g) True. Take a basis $\beta_i$ of each $T$–invariant subspace $W_i$ and consider the basis $\cup_i \beta_i$ for the whole space.

5.4.3 (a) Since $T : V \to V$ is linear, clearly $T(\{0\}) = \{0\}$ and $T(V) \subseteq V$.

(b) Given $x \in N(T)$ we have $Tx = 0 \in N(T)$. Given $y \in R(T)$ we have $Ty \in R(T)$ by definition of the range.

(c) Given $v \in E_\lambda$, we have $Tv = \lambda v \in E_\lambda$.

5.4.18 (a) Given any $x \in W$, we can find scalars $c_0, \ldots, c_k$ so that $x = c_0 v + c_1 Tv + \cdots + c_k T^k v$. Then

$$Tx = c_0 Tv + c_1 T^2 v + \cdots + c_k T^{k+1} v,$$

which is also a linear combination of vectors of the form $T^j v$. Hence $Tx \in W$.

(b) Let $S$ be a $T$–invariant subspace of $V$ containing $v$. An inductive argument shows that every iterate $T^j v$ is in $S$; indeed, $T^0 v = v \in S$ and whenever $T^j v \in S$ we must also have $T^{j+1} v = T(T^j v) \in S$ by virtue of $T$–invariance. Since $S$ is a space, it must contain the span of the iterates $T^j v$, which is precisely the space $W$.

5.4.21 Let $V$ be a two–dimensional space and $T : V \to V$. There are two possibilities that can occur: either $\{x, Tx\}$ is linearly independent for some vector $x \in V$, or $\{x, Tx\}$ is always linearly dependent. In the former situation we'd have that $V$ is the $T$–cyclic space generated by $x$, so we'd be done. Hence we assume that $\{x, Tx\}$ is linearly dependent for every vector $x \in V$. We will show in this case that $T = cI$ for some scalar $c$; that is, for some fixed scalar $c$ every vector $x \in V$ satisfies $Tx = cx$. This is clearly true for $x = 0$, so from here on we only consider nonzero vectors $x$.

Given a nonzero vector $x \in V$, the linear dependence of the set $\{x, Tx\}$ means we can find scalars $c_1$ and $c_2$ — not both zero — so that

$$c_1 x + c_2 Tx = 0.$$

If $c_2 = 0$ then $c_1 \neq 0$ and we must have $x = 0$. Since we wanted to exclude this case, we cannot have $c_2 = 0$. Dividing by $c_2$, we see that

$$Tx = -(c_1/c_2)x.$$

Rewriting $-c_1/c_2 = c$, we find that every vector $x \in V$ satisfies an eigenvalue equation $Tx = cx$. We do not yet know, however, that the same scalar $c$ works for all vectors.

Consider nonzero vectors $x, y \in V$ and find scalars $a, b$ so that $Tx = ax$ and $Ty = by$. If we show that $a = b$, then we're done. To this end, we can assume that $x$ and $y$ are not scalar multiples of each other; that is, assume that $\{x, y\}$ is linearly independent. The vector $x + y$ also satisfies an eigenvalue equation of the form $T(x + y) = c(x + y)$, but linearity yields

$$c(x + y) = T(x + y) = Tx + Ty = ax + by,$$

so that $(c - a)x + (c - b)y = 0$. The linear independence of $\{x, y\}$ implies that $a = b$, as desired.

6.1.10 Assuming that $x$ and $y$ are orthogonal vectors in an inner product space $V$, we have

$$\|x + y\|^2 = \langle x + y, x + y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \langle x, x\rangle + \langle y, y\rangle = \|x\|^2 + \|y\|^2.$$

The standard inner product in $\mathbb{R}^2$ yields the Pythagorean theorem.

6.1.17 Let $T$ be a linear operator on an inner product space $V$ satisfying $\|Tx\| = \|x\|$ for all $x \in V$ (such an operator is typically called an isometry). If $x \in V$ is chosen so that $Tx = 0$, then $\|x\| = \|Tx\| = 0$, so $x = 0$ as well. That is, the null space of $T$ is trivial, so $T$ is injective.

## Homework 4

6.2.6 Given a vector $x \notin W$ we can use theorem 6.6 to write $x = w + z$, with $w \in W$ and $z \in W^\perp$. If $z$ were zero then we would have $x = w \in W$, so $z \neq 0$. Notice that $z \in W^\perp$ and

$$\langle x, z\rangle = \langle w + z, z\rangle = \langle w, z\rangle + \langle z, z\rangle = 0 + \|z\|^2 \neq 0.$$

6.2.11 Given $i, j \in \{1, 2, \ldots, n\}$ the component formula for matrix multiplication gives

$$(AA^*)_{ij} = \sum_{k=1}^{n} A_{ik}(A^*)_{kj} = \sum_{k=1}^{n} A_{ik}\overline{A}_{jk}.$$

This last expression is the inner product of rows $i$ and $j$; thus if the rows of $AA^*$ are orthonormal, we have $(AA^*)_{ij} = 0$ when $i \neq j$ and $(AA^*)_{ii} = 1$. That is, $AA^* = I$. Conversely if $AA^* = I$ then the rows are orthonormal by the same argument in reverse.

6.2.13 (a) Assume that $S_0 \subseteq S$ and let $y \in S^\perp$. Given an arbitrary $x \in S_0$ we have $x \in S$, so $\langle x, y\rangle = 0$. Thus $y \in S_0^\perp$ and we conclude $S^\perp \subseteq S_0^\perp$.

(b) Given $x \in S$ note that $x$ is orthogonal to every element of $S^\perp$. But then $x \in (S^\perp)^\perp$. Since the orthogonal complement of any set is a space, this implies $\text{span}(S) \subseteq (S^\perp)^\perp$.

(c) It remains to show that $(W^\perp)^\perp \subseteq W$. Suppose that $x \notin W$; by problem 6.2.6 there exists $y \in W^\perp$ with $\langle x, y\rangle \neq 0$. That is, $x$ is not orthogonal to some element of $W^\perp$ and $x \notin (W^\perp)^\perp$.

(d) By theorem 6.6 any vector in $V$ can be written as a sum of the form $w + z$, with $w \in W$ and $z \in W^\perp$. Hence $V = W + W^\perp$. Suppose that $y \in W \cap W^\perp$. Since $y \in W^\perp$, it must be orthogonal to every element of $W$; in particular $\langle y, y\rangle = 0$. This implies $y = 0$, so $W \cap W^\perp = \{0\}$. We conclude that $V = W \oplus W^\perp$.

6.3.7 Consider $\mathbb{R}^2$ with the standard inner product and standard basis $\{e_1, e_2\}$. Define the linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(e_1) = 0$ and $T(e_2) = e_1$. Notice that $e_2 \notin N(T)$; we'll show that $e_2 \in N(T^*)$. We compute $\langle e_1, T^*e_2\rangle = \langle Te_1, e_2\rangle = 0$ and $\langle e_2, T^*e_2\rangle = \langle Te_2, e_2\rangle = \langle e_1, e_2\rangle = 0$. Since $T^*e_2$ is orthogonal to both vectors in the basis, $T^*e_2 = 0$. Thus $N(T) \neq N(T^*)$.

6.3.9 Let $x \in V$ be arbitrary; we want to show that $T^*x - Tx = 0$. Since $V = W \oplus W^\perp$ by assumption, we can write $T^*x - Tx = w + z$ with $w \in W$ and $z \in W^\perp$. Note that

$$\|T^*x - Tx\|^2 = \langle T^*x - Tx, T^*x - Tx\rangle = \langle T^*x - Tx, w\rangle + \langle T^*x - Tx, z\rangle,$$

so we need only show that each inner product on the right vanishes. For the first product, we make two observations: $w \in W$ implies $Tw = w$ and $x - Tx \in W^\perp$. With these in mind we have

$$\langle T^*x - Tx, w\rangle = \langle T^*x, w\rangle - \langle Tx, w\rangle = \langle x, Tw\rangle - \langle Tx, w\rangle = \langle x, w\rangle - \langle Tx, w\rangle = \langle x - Tx, w\rangle = 0.$$

For the second product, note that since $z \in W^\perp$ we have both $Tz = 0$ and $\langle Tx, z\rangle = 0$. This gives

$$\langle T^*x - Tx, z\rangle = \langle T^*x, z\rangle - \langle Tx, z\rangle = \langle x, Tz\rangle - 0 = 0.$$

We conclude that $T^*x - Tx = 0$; as $x$ was arbitrary, $T = T^*$.

6.3.12 (a) Let $x \in R(T^*)^\perp$. Then for any vector $y \in V$ we have $\langle x, T^*y\rangle = 0$. Choosing $y = Tx$ gives

$$0 = \langle x, T^*Tx\rangle = \langle Tx, Tx\rangle = \|Tx\|^2,$$

so $Tx = 0$. That is, $x \in N(T)$ and $R(T^*)^\perp \subseteq N(T)$.
Now suppose that $x \in N(T)$ and $T^*y \in R(T^*)$ is arbitrary. Then

$$\langle x, T^*y\rangle = \langle Tx, y\rangle = 0,$$

so $x \in R(T^*)^\perp$. We conclude that $N(T) \subseteq R(T^*)^\perp$, whence the spaces are equal.

(b) From exercise 6.2.13 we know that in a finite–dimensional space, $(W^\perp)^\perp = W$ for any subspace $W$. Using the previous part gives

$$N(T)^\perp = (R(T^*)^\perp)^\perp = R(T^*),$$

as desired.

# Homework 5

6.4.1 (a) True.

(b) False. Consider the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(e_1) = 0$ and $T(e_2) = e_1$. You can show that $T^*(e_1) = e_2$ and $T(e_2) = 0$. That is, $e_1$ is an eigenvector of $T$ with eigenvalue 0, but not for $T^*$.

(c) False. This only holds for orthonormal bases $\beta$.

(d) True.

(e) True.

(f) True.

(g) False; if the underlying field is real, self–adjointness is needed.

(h) True.

6.4.7 (a) Let $u, v \in W$. Then
$$\langle T_W u, v\rangle_W = \langle Tu, v\rangle = \langle u, Tv\rangle = \langle u, T_W v\rangle_W.$$
Hence $T_W$ is self–adjoint.

(b) Let $v \in W^\perp$. Given $u \in W$ we have
$$\langle T^* v, u\rangle = \langle v, Tu\rangle = 0,$$
since $Tu \in W$. Thus $T^* v \in W^\perp$ and $W^\perp$ is $T^*$–invariant.

(c) Let $u, v \in W$. Then
$$\langle u, (T_W)^* v\rangle_W = \langle T_W u, v\rangle_W = \langle Tu, v\rangle = \langle u, T^* v\rangle = \langle u, (T^*)_W v\rangle_W.$$
Thus $[(T_W)^* - (T^*)_W]v$ an element of $W \cap W^\perp = \{0\}$.

(d) Using the previous part, $T_W(T_W)^* = T_W(T^*)_W = (TT^*)_W$. Similarly $(T_W)^* T_W = (T^*T)_W$. The result follows from the assumption that $TT^* = T^*T$.

6.4.11 (a) Let $x \in V$ and note that
$$\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}.$$
Hence $\langle Tx, x\rangle$ equals its complex conjugate and is real.

(b) Let $x, y \in V$ and note that
$$\begin{aligned} 0 &= \langle T(x + y), x + y\rangle \\ &= \langle Tx, x\rangle + \langle Tx, y\rangle + \langle Ty, x\rangle + \langle Ty, y\rangle \\ &= \langle Tx, y\rangle + \langle y, Tx\rangle \\ &= 2\mathrm{Re}\langle Tx, y\rangle, \end{aligned}$$
so $\langle Tx, x\rangle$ is purely imaginary. However we can also expand
$$\begin{aligned} 0 &= \langle T(x + iy), x + iy\rangle \\ &= \langle Tx, x\rangle + \langle Tx, iy\rangle + \langle Tiy, x\rangle + \langle Tiy, iy\rangle \\ &= -i\langle Tx, y\rangle + i\langle y, Tx\rangle \\ &= 2\mathrm{Im}\langle Tx, y\rangle, \end{aligned}$$
whence $\langle Tx, y\rangle$ has zero imaginary part as well. Thus $\langle Tx, y\rangle = 0$ for any $x, y$; choosing $y = Tx$ shows $Tx = 0$. As $x$ was arbitrary, $T = 0$.

(c) Let $x \in V$ and compute

$$\langle (T - T^*)x, x\rangle = \langle Tx, x\rangle - \langle T^*x, x\rangle = \langle Tx, x\rangle - \langle x, Tx\rangle.$$

Since $\langle Tx, x\rangle \in \mathbb{R}$, this last expression is zero. By the previous part, knowing that $\langle (T - T^*)x, x\rangle = 0$ for all $x$ implies that $T = T^*$.

6.5.6 Let $f, g \in V$ and write

$$\langle Tf, g\rangle = \int_0^1 (Tf)(t)\overline{g(t)}\, dt = \int_0^1 h(t)f(t)\overline{g(t)}\, dt = \int_0^1 f(t)\overline{\overline{h(t)}g(t)}\, dt = \langle f, \overline{h}g\rangle.$$

From this we see that $T^*g = \overline{h}g$. Then $TT^*f = T^*Tf = |h|^2 f$; this is the identity if and only if $|h| = 1$.

6.5.7 Choose an orthonormal ordered basis so that

$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Since $T$ is unitary, each $\lambda_k$ has modulus 1 and hence has a square root of modulus 1 (for those unfamiliar with complex arithmetic, a number has absolute value 1 if and only if it can be written as $e^{it}$ with $t \in \mathbb{R}$; then $e^{it/2}$ has modulus one and squares to $e^{it}$). Defining $U$ as

$$[U]_\beta = \begin{pmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \sqrt{\lambda_n} \end{pmatrix}$$

gives the required square root of $T$.

6.5.10 We need the fact that similar matrices have equal traces. With this in hand, we can diagonalize $A$; in diagonal form the nonzero entries are the eignevalues and clearly $\mathrm{tr}(A)$ is their sum.

Furthermore, having diagonalized $A$ we have that

$$PAP^* = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \quad \text{and} \quad PA^*P^* = \begin{pmatrix} \overline{\lambda_1} & 0 & \cdots & 0 \\ 0 & \overline{\lambda_2} & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \overline{\lambda_n} \end{pmatrix}.$$

From this it follows that

$$PA^*AP^* = \begin{pmatrix} |\lambda_1|^2 & 0 & \cdots & 0 \\ 0 & |\lambda_2|^2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & |\lambda_n|^2 \end{pmatrix}.$$

Clearly then $\mathrm{tr}(A^*A) = \sum |\lambda_k|^2$.

# Homework 6

6.6.1 (a) False. Only orthogonal projections are self–adjoint.

(b) True.

(c) True. (The Spectral Theorem)

(d) False. Only true for orthogonal projections.

(e) False. Most projections aren't invertible, let alone unitary.

6.6.4 Since $W$ is finite dimensional, we have that $V = W \oplus W^{\perp}$. By definition of $T$, whenever $x \in W$ and $y \in W^{\perp}$, we have $T(x + y) = x$. But then $(I - T)(x + y) = y$. The finite dimensionality of $W$ implies that $(W^{\perp})^{\perp} = W$, so $I - T$ is an orthogonal projection derived from the direct sum $V = W^{\perp} \oplus (W^{\perp})^{\perp}$.

6.6.6 We've seen many times that for any projection $T$, the space decomposes as $V = R(T) \oplus N(T)$. Assume that $T$ is normal; to show that $T$ is an orthogonal projection, we need only show that $N(T) = R(T)^{\perp}$. Let $x \in N(T)$ and consider arbitrary $y \in R(T)$. Then $y = Ty$ and

$$\langle x, y \rangle = \langle x, Ty \rangle = \langle T^* x, y \rangle = 0,$$

since $Tx = 0x \Leftrightarrow T^* x = \overline{0} x = 0$.

Next assume that $v \in R(T)^{\perp}$. Then

$$\langle Tv, Tv \rangle = \langle v, T^* T v \rangle = \langle v, T T^* v \rangle = 0,$$

since we've previously shown that $R(T)^{\perp} = N(T^*)$. Hence $Tv = 0$ and $v \in N(T)$.

# Homework 7

6.7.13 Assume that $A$ is positive semidefinite. The eigenvalues of $A^*A$ are $\sigma_i^2$, where $\sigma_i$ denotes the singular values of $A$. Part of the definition of positive definiteness requires $A$ to be self–adjoint, so $A^2$ has eigenvalues $\sigma_i^2$. If $\lambda_i$ denotes an eigenvalue of $A$, we know that $\lambda_i^2$ is an eigenvalue of $A^2$; after rearranging the lists, we have $\lambda_i^2 = \sigma_i^2$ for each $i$. Finally, the eigenvalues and singular values of a positive semidefinite matrix are nonnegative, so we conclude that $\sigma_i = \lambda_i$ for each $i$.

6.7.15 (a) ($\Rightarrow$) Assume that $A$ is normal, and consider the polar decomposition $A = WP$ with $W$ unitary and $P$ positive semidefinite. Then

$$P^* W^* W P = (WP)^* W P = A^* A = A A^* = WP(WP)^* = W P P^* W^*.$$

By definition we assume that positive operators are self–adjoint, so $P^* = P$. Furthermore, $WW^* = W^*W = I$, so we have

$$P^2 = W P^2 W^*.$$

Multiplying on the right by $W$ gives the result.

($\Leftarrow$) Assume that $WP^2 = P^2 W$ and note that

$$A^* A W = P^* W^* W P W = P^2 W = W P^2 = W P^2 W^* W = A A^* W.$$

Multiplying by $W^*$ on the right gives $AA^* = A^*A$.

(b) ($\Rightarrow$) Assume that $A$ is normal. By the previous part, $WP^2 = P^2 W$, which rearranges into

$$P^2 = W^* P^2 W = W^* P (W W^*) P W = (W^* P W)^2.$$

Since $W^*PW$ is unitarily equivalent to $P$, it is also positive semidefinite. Positive semidefinite operators admit square roots, so the above equation becomes $P = W^*PW$, which is equivalent to $WP = PW$.

($\Leftarrow$) Assume that $WP = PW$. Then $WP^2 = PWP = P^2 W$, so $A$ is normal by the previous part.

6.8.7 (a) Given $H \in \mathcal{B}(W)$, the domain of $\widehat{T}H$ is certainly $V \times V$. For bilinearity, we check the first argument:

$$\begin{aligned}\widehat{T}H(ax+y,z) &= H(T(ax+y),Tz) = H(aTx+Ty,Tz) \\ &= aH(Tx,Tz) + H(Ty,Tz) \\ &= a\widehat{T}H(x,z) + \widehat{T}H(y,z)\end{aligned}$$

holds for any $x, y, z \in V$ and scalar $a$. Hence $\widehat{T}H$ is linear in the first argument; the same reasoning shows $\widehat{T}H$ to be linear in its second argument, so we conclude that $\widehat{T}H \in \mathcal{B}(V)$.

(b) This is similar to the previous part. Given bilinear forms $H, J \in \mathcal{B}(W)$ and a scalar $a$, we have

$$\begin{aligned}\widehat{T}(aH+J)(x,y) &= (aH+J)(Tx,Ty) = aH(Tx,Ty) + J(Tx,Ty) \\ &= a\widehat{T}H(x,y) + \widehat{T}J(x,y)\end{aligned}$$

for any $x, y \in V$. Hence $\widehat{T}$ is linear.

(c) We could show that $\widehat{T}$ is bijective, but instead we can construct an obvious inverse map. Note that

$$\begin{aligned}\widehat{T}\widehat{T^{-1}}H(x,y) &= H(TT^{-1}x, TT^{-1}y) = H(x,y) \\ \widehat{T^{-1}}\widehat{T}H(x,y) &= H(T^{-1}Tx, T^{-1}Ty) = H(x,y),\end{aligned}$$

so $\widehat{T^{-1}}$ is an inverse map of $\widehat{T}$, proving $\widehat{T}$ to be invertible.