Math 108B Selected Homework Solutions

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Part I

1. A linear operator T on a space V is a projection if $T^2 = T$.
   True

2. The eigenvalues of a diagonalizable operator are nonzero.
   False; the zero operator is diagonal and has all eigenvalues zero

3. $\langle z, w \rangle = z\overline{w}$ defines an inner product on $\mathbb{C}$.
   True

4. Every linear operator on a finite dimensional space has an adjoint.
   True; T^* can be constructed via the rule $\langle Tx, y \rangle = \langle x, T^* y \rangle$

5. Unitary operators are normal.
   True; a unitary operator has inverse equal to its adjoint, and an invertible operator commute with its inverse

Part II

1. If T is a linear operator on V with characteristic polynomial $p(t) = 2t^5 + t^3 - 1$, prove that $T^{-1} = 2T^4 + T^2$.

   Proof. By the Cayley–Hamilton theorem, $p(T) = 0$; that is, $2T^5 + T^3 - I = 0$. Since $p(0) = -1 \neq 0$, the operator T is invertible. Multiplying the equation by T^{-1} gives

   $$0 = T^{-1}(2T^5 + T^3 - I) = 2T^4 + T^2 - T^{-1},$$

   which rearranges into the result. □

2. Let T be a linear operator on a finite dimensional inner product space V with $\|Tx\| = \|x\|$ for all $x \in V$. Prove that T is an isomorphism.

   Proof. Suppose that $x \in N(T)$. Then $\|x\| = \|Tx\| = 0$, whence $x = 0$. Therefore the null space of T is trivial and T is injective; from the finite dimensionality of V, this is enough to conclude that T is an isomorphism. □

3. Let T be a linear operator on a finite dimensional inner product space V with W a T–invarint subspace. Prove that $W^\perp$ is T–invariant or give a counterexample.

Proof. The statement is false. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then $T(1,0) = (1,0) \in W = \text{span}(\{(1,0)\})$, so the x–axis is T–invariant. However $(0,1) \in W^\perp$, yet $T(0,1) = (1,1) \notin W^\perp$. So $W^\perp$ is not T–invariant. □

4. Let T be a linear operator on a finite dimensional inner product space V having an eigenvector with eigenvalue λ. Prove that T^* has an eigenvector with eigenvalue $\overline{\lambda}$.

   Proof. The characteristic polynomials of T and T^* are given by

   $$\det(T - \lambda I) = \det((T - \lambda I)^*) = \det(T^* - \overline{\lambda} I).$$

   Thus if λ is a root of $\det(T - \lambda I)$ (that is, an eigenvalue of T), then $\overline{\lambda}$ is a root of $\det(T - \overline{\lambda} I)$. □

5. Determine whether T on $V = M_{2\times 2}(\mathbb{R})$ defined by $T(A) = A^t$ is normal, self–adjoint, unitary, orthogonal, or none of these.

   Proof. The inner product on V is $\langle A, B\rangle = \text{tr}(B^*A) = \text{tr}(B^tA)$, since the undrlying field is real. Recall two properties of trace: for any matrices M, N we have $\text{tr}(M^t) = \text{tr}(M)$ and $\text{tr}(MN) = \text{tr}(NM)$. Thus we have

   $$\langle TA, B\rangle = \text{tr}(B^tA^t) = \text{tr}(A^tB^t) = \text{tr}((BA)^t) = \text{tr}(BA) = \text{tr}((B^t)^tA) = \langle A, TB\rangle.$$

   Thus T is self–adjoint, hence normal. Furthermore, $T^2(A) = (A^t)^t = A$, so T is its own inverse. As T is its own adjoint as well, its adjoint is its inverse. That is, T is unitary. Finally, since the underlying field is real we have $T = T^* = T^t$ and T is orthogonal as well. □

6. Let T be a linear operator on a finite dimensional inner product space V with $\|Tx\| = \|x\|$ for all $x \in V$. Prove that $T^* = T^{-1}$.

   Proof. From a previous problem on the exam, T is invertible. Once we know that $T^*T = I$, the other order follows (because on a finite dimensional space, a one–sided inverse is two–sided). Let $U = I - T^*T$; it suffices to show that $Ux = 0$ for arbitrary $x \in V$. We can write

   $$\langle Ux, x\rangle = \langle x - T^*Tx, x\rangle = \langle x, x\rangle - \langle x, T^*Tx\rangle = \|x\|^2 - \langle Tx, Tx\rangle = 0,$$

   by our assumption on the preservation of norm. Note that U is self adjoint: $U^* = (I - T^*T)^* = I^* - (T^*T)^* = I - T^*(T^*)^* = I - T^*T = U$. Since the underlying vector space is finite dimensional, U is diagonalizable. To show that $U = 0$, it suffices to show that each eigenvalue is zero (for then its matrix in an appropriate basis will consist entirely of zeroes). This is easy; given an eigenvalue λ of U with eigenvector $v \neq 0$, we have

   $$\lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle Uv, v\rangle = 0.$$

   Since $\langle v, v\rangle \neq 0$, this means $\lambda = 0$. Thus each eigenvalue is zero, $U = 0$ as well, and $T^*T = I$. □

7. Let T be a normal operator on a finite dimensional inner product space V with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Let W_i be the eigenspace corresponding to the eigenvalue λ_i. Let T_i be the orthogonal projection onto W_i. Prove that $I = T_1 + \cdots + T_k$ and $T = \lambda_1 T_1 + \cdots \lambda_k T_k$.

Proof. Assuming that the underlying field is complex, T is diagonalizable. That is, any $v \in V$ can be written as

$$v = x_1 + x_2 + \cdots + x_k,$$

with $x_i \in W_i$ for each $1 \leq i \leq k$. For a normal operator, distinct eigenspaces are orthogonal, so the orthogonal projections onto each eigenspace satisfy

$$T_i v = T_i(x_1 + x_2 + \cdots + x_k) = x_i,$$

which implies that

$$v = T_1 v + T_2 v + \cdots + T_k v = (T_1 + \cdots + T_k)v.$$

As v was arbitrary, we have $I = T_1 + \cdots T_k$. Finally, we also have that

$$Tv = Tx_1 + Tx_2 + \cdots + Tx_k = \lambda_1 x_1 + \cdots + \lambda_k x_k = \lambda_1 T_1 v + \cdots \lambda_k T_k v = (\lambda_1 T_1 + \cdots + \lambda_k T_k)v.$$

As v is arbitrary, $T = \lambda_1 T_1 + \cdots + \lambda_k T_k$. □

8. Consider $C[-1,1]$, the vector space of continuous, real–valued functions on $[-1,1]$, equipped with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,\frac{dx}{\sqrt{1-x^2}}.$$

Let $S = \{1, x\}$. Show that S is orthogonal and find the orthogonal projection of $f(x) = x^2$ onto $\text{span}(S)$.

Proof. The set S has only two elements, so we need only compute a single inner product to check orthogonality:

$$\langle 1, x \rangle = \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\,dx = 0,$$

since this is the integral of an odd function over an interval symmetric about zero.

To orthogonally project x^2 onto $\text{span}(S)$, we want to find scalars c_1, c_2 so that

$$x^2 = c_1 + c_2 x + g(x),$$

with $g(x) \in S^\perp$. Taking an inner product against 1 gives

$$\langle 1, x^2 \rangle = c_1 \langle 1, 1 \rangle + c_2 \langle 1, x \rangle + \langle 1, g(x) \rangle = c_1 \langle 1, 1 \rangle,$$

since $\langle 1, x \rangle = \langle 1, g \rangle = 0$. We compute

$$\langle 1, x^2 \rangle = \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_{-\pi/2}^{\pi/2} \sin^2\theta\,d\theta = \frac{\pi}{2}$$

$$\langle 1, 1 \rangle = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \int_{-\pi/2}^{\pi/2} d\theta = \pi.$$

From this we find $c_1 = 1/2$. Furthermore

$$c_2 = \frac{\langle x, x^2 \rangle}{\langle x, x \rangle} = \frac{1}{\langle x, x \rangle} \int_{-1}^{1} \frac{x^3}{\sqrt{1-x^2}}\,dx = 0,$$

as once again we are integrating an odd function over a symmetric interval. Thus our orthogonal projection of x^2 onto $\text{span}(S)$ is the constant function $h(x) = 1/2$. □