Math 118B Solutions

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Homework Problems

1. Let (X_i, d_i), $1 \leq i \leq n$, be finitely many metric spaces. Construct a metric on the product space $X = X_1 \times \cdots \times X_n$.

   Proof. Denote points in X as $x = (x_1, x_2, \ldots, x_n)$. Given $x, y \in X$ define $d(x, y) = d_1(x_1, y_1) + \cdots + d(x_n, y_n)$. Then clearly $d(x, y) \geq 0$ and $d(x, y) = d(y, x)$. Furthermore, if $d(x, y) = 0$ then each $d_i(x_i, y_i) = 0$. That is, each $x_i = y_i$ and $x = y$. Finally, given $x, y, z \in X$ we have

   $$d(x, z) = \sum_{i=1}^{n} d_i(x_i, z_i) \leq \sum_{i=1}^{n} d_i(x_i, y_i) + d_i(y_i, z_i) = d(x, y) + d(y, z).$$

   Hence d is a metric on X. □

2. Let (X, d) be a metric space. Prove that

   $$\delta(x, y) = \frac{d(x, y)}{1 + d(x, y)}; \qquad x, y \in X$$

   is also a metric on X.

   Proof. Clearly we have that $\delta(x, y) = \delta(y, x)$ and $\delta(x, y) \geq 0$. If $\delta(x, y) = 0$, then $d(x, y) = 0$ and $x = y$. Finally, given $x, y, z \in X$ we have

   $$\begin{aligned} \delta(x, z) &= \frac{d(x, z)}{1 + d(x, z)} \\ &= 1 - \frac{1}{1 + d(x, z)} \\ &\leq 1 - \frac{1}{1 + d(x, y) + d(y, z)} \\ &= \frac{d(x, y) + d(y, z)}{1 + d(x, y) + d(y, z)} \\ &= \frac{d(x, y)}{1 + d(x, y) + d(y, z)} + \frac{d(y, z)}{1 + d(x, y) + d(y, z)} \\ &\leq \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} \\ &= \delta(x, y) + \delta(y, z). \end{aligned}$$

   Hence δ is a metric on X. □

3. Prove that the space ℓ_∞ of bounded sequences is complete with respect to the $\| \cdot \|_\infty$ distance. Let c, c_0 denote the subspaces of ℓ_∞ of convergent, respectively, convergent to zero, sequences. Prove that c, c_0 are closed subspaces of ℓ_∞.

Proof. This is a bit of a notational challenge. Let $x_1, x_2, x_3 \ldots$ be a sequence of points within ℓ_∞, each of which is *itself a sequence of real numbers.* That is, for each $i \geq 1$ we have that

$$x_i = (x_{i1}, x_{i2}, x_{i3}, \ldots).$$

Suppose that the sequence (x_i) is Cauchy in ℓ_∞. Then for each $i, m, n \geq 1$ we have

$$|x_{im} - x_{in}| \leq \sup_i |x_{im} - x_{in}| = \|x_m - x_n\|_\infty \to 0$$

as $m, n \to \infty$. That is, each sequence of real numbers $x_{i1}, x_{i2}, \ldots$ is Cauchy in $\mathbb{R}$, hence convergent. Let $y_i = \lim x_{in}$ and consider the sequence $y = (y_i)$.

Let $\epsilon > 0$. Find N (depending only upon ϵ) so that for all $m, n \geq N$ we have $\|x_n - x_m\|_\infty < \epsilon/2$. For each k this implies that $|x_{nk} - x_{mk}| < \epsilon/2$. If we take $m \to \infty$ then $x_{mk} \to y_k$ and we have that $|x_{nk} - y_k| \leq \epsilon/2$ for all $k \geq 1$. Taking the supremum over k gives $\|x_n - y\|_\infty \leq \epsilon/2 < \epsilon$. This shows that $\|x_n - y\|_\infty \to 0$ as $n \to \infty$. We should also verify that $y \in \ell_\infty$; if n is such that $\|x_n - y\|_\infty < 1$ then

$$\|y\|_\infty = \|x_n + (x_n - y)\|_\infty < \|x_n\|_\infty + 1 < \infty,$$

so indeed $y \in \ell_\infty$.

Now we turn our attention to c and c_0; since convergent sequences are bounded we have that $c, c_0 \subset \ell_\infty$. It is clear that c, c_0 are both subspaces, so we'll only show that they are both closed. Suppose that (x_n) is a sequence in c which converges to some point $y \in \ell_\infty$. We wish to show that $y \in c$. Let $\epsilon > 0$ and find n so that $\|x_n - y\|_\infty < \epsilon/3$. Now find M so that for all $j, k \geq M$ we have $|x_{nj} - x_{nk}| < \epsilon/3$. For all such j, k we have

$$\begin{aligned}
|y_j - y_k| &\leq |y_j - x_{nj}| + |x_{nj} - x_{nk}| + |x_{nk} - y_k| \\
&\leq \|y - x_n\|_\infty + |x_{nj} - x_{nk}| + \|y - x_k\|_\infty \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

So y is a Cauchy sequence of real numbers, hence convergent. That is, $y \in c$.

Now suppose that (x_n) is a sequence in c_0 which converges to some $y \in \ell_\infty$. Since $c_0 \subset c$, the above reasoning shows that $y \in c$; we need only show that $\lim y = 0$. Let $\epsilon > 0$ and take n so that $\|y - x_n\|_\infty < \epsilon/2$. Then for each k we have $|y_k - x_{nk}| < \epsilon/2$. Find M so that for all $k \geq M$ we have $|x_{nk}| < \epsilon/2$; for all such k we have

$$|y_k| \leq |y_k - x_{nk}| + |x_{nk}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So $\lim y = 0$ as desired. □

4. Describe the intervals $[0, a] \subset \mathbb{R}$ such that the function $f : [0, a] \to [0, a]$, $f(x) = \sin x$, is contractive.

Proof. Let $a > 0$. Suppose there exists a $c < 1$ so that for all $x, y \in [0, a]$,

$$|\sin x - \sin y| \leq c|x - y|.$$

Let $x \in (0, a]$ and note that

$$\frac{|\sin x - \sin 0|}{|x - 0|} < c.$$

But if we take $x \to 0$, we have that $1 \leq c$, a contradiction. Upon no such interval is the sine function contractive. □

5. Let A be a bounded subset of $C[0, 1]$. Prove that the set of functions

$$F(x) = \int_0^x f(t)\, dt, \qquad f \in A,$$

is equicontinuous in $C[0, 1]$.

Proof. For convenience denote

$$\mathcal{F} = \left\{ \int_0^x f(t)\, dt : f \in A \right\}.$$

Choose $M > 0$ so that for each $f \in A$ we have $\|f\|_\infty \le M$. Given $\epsilon > 0$ set $\delta = \epsilon/M$. Whenever $x, y \in [0,1]$ are such that $|x - y| < \delta$ and $f \in A$, it follows that

$$\left| \int_0^x f(t)\, dt - \int_0^y f(t)\, dt \right| = \left| \int_y^x f(t)\, dt \right| \le M|x - y| < M\delta = \epsilon,$$

so we conclude that $\mathcal{F}$ is equicontinuous. □

6. Is the sequence of functions $\sin(nx)$, $n \ge 1$, equicontinuous in $C[0,1]$?

Proof. Suppose that the sequence is equicontinuous. In particular, there exists a δ so that for any n and $x, y \in \mathbb{R}$ with $|x - y| < \delta$,

$$|\sin nx - \sin ny| < 1.$$

Let n be so large that $\pi/2n < \delta$. Then $|\pi/2n - 0| < \delta$, yet

$$|\sin(n \cdot \pi/2n) - \sin(n \cdot 0)| = 1,$$

a contradiction. The family is not equicontinuous. □

7. Start with $P_0 = 0$ and define for $n \ge 0$

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}.$$

Prove that $P_n(x) \to |x|$ uniformly on $[-1, 1]$.

Proof. We start by proving that $0 \le P_n(x) \le |x|$ for all $x \in [-1,1]$ and $n \ge 0$. This is true for $P_0(x) = 0$, so assume it true for some $n \ge 0$. Then $x^2 - P_n^2(x) \ge 0$ and $P_n(x) \ge 0$ implies that $P_{n+1}(x) \ge 0$. Further, notice that

$$|x| - P_{n+1}(x) = [|x| - P_n(x)] \left[1 - \frac{|x| + P_n(x)}{2} \right]. \tag{1}$$

Each factor on the right is positive since $P_n \le |x|$ and $(|x| + P_n(x))/2 \le |x| \le 1$; thus $P_{n+1}(x) \le |x|$. By induction $0 \le P_n(x) \le |x|$ for all $x \in [-1,1]$ and $n \ge 0$.

FIRST PROOF Define the functions $g_n(x) = |x| - P_n(x)$; we want to show that $g_n \to 0$ uniformly. Notice that

$$g_{n+1}(x) = g_n(x) \left[1 - \frac{|x| + P_n(x)}{2} \right] \le g_n(x) \left[1 - \frac{|x|}{2} \right].$$

Since $g_0(x) = |x|$ we find that

$$g_n(x) \le |x| \left(1 - \frac{|x|}{2} \right)^n.$$

We wish to bound the right side from above uniformly in x. It suffices to consider $x \ge 0$; define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = x \left(1 - \frac{x}{2} \right)^n.$$

Since f is continuous on a compact interval, it attains an absolute maximum. To find candidates for the maximum, we need

$$\begin{aligned} 0 &= \frac{d}{dx} \ln f(x) \\ &= \frac{1}{x} - \frac{n}{2 - x} \\ &= \frac{2 - x(n+1)}{x(2 - x)}. \end{aligned}$$

The only relative maximum of f occurs at $x = 2/(n+1)$. Since $f(0) = 0$ and $f(1) = 2^{-n}$, the absolute maximum is at $x = 2/(n+1)$. Therefore

$$g_n(x) \le |x|\left(1 - \frac{|x|}{2}\right)^n \le f\left(\frac{2}{n+1}\right) = \frac{2}{n+1}\left(1 - \frac{1}{n+1}\right)^n < \frac{2}{n+1}.$$

Thus $\|g_n\|_\infty \le 2/(n+1)$ and $g_n \to 0$ uniformly, as desired.

ALTERNATIVE PROOF Once we know that $0 \le P_n(x) \le |x|$ we see that

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2} \ge P_n(x),$$

so for each x the sequence $(P_n(x))$ is monotone and bounded by $|x|$, hence pointwise convergent. This limit function must be $|x|$; if we denote $P_n(x) \to L$ then

$$L = L + \frac{x^2 - L^2}{2},$$

so $L = |x|$ because $L \ge 0$. Uniform convergence now follows from Dini's theorem:

Theorem (Dini)**.** *Let X be a compact metric space and suppose that*

$$f_1 \ge f_2 \ge f_3 \ge \cdots$$

are continuous real-valued functions which converge pointwise to a continuous function f. Then the convergence is uniform.

Proof of Dini's theorem. If we consider $f_n - f$, we have a monotone sequence of continuous functions which converge pointwise to 0 (from above); henceforth we'll assume the pointwise limit is 0. If f_n does not converge uniformly to 0, there exists $\epsilon > 0$ and a subsequence (f_{n_k}) so that $\|f_{n_k}\|_\infty \ge 2\epsilon$ for all k. That is, we can find a sequence of points (x_{n_k}) in X so that $f_{n_k}(x_{n_k}) \ge \epsilon$ for all k. Since X is compact there is a convergent subsequence of (x_{n_k}); for simplicity, we'll not denote a new sequence and instead assume that $x_{n_k} \to x \in X$. Here's where we use monotonicity: fix j and note that for any $k > j$

$$\epsilon \le f_{n_k}(x_{n_k}) \le f_{n_j}(x_{n_k}).$$

If we take $k \to \infty$ then $f_{n_j}(x_{n_k}) \to f_{n_j}(x)$ by continuity; thus $f_{n_j}(x) \ge \epsilon$ for arbitrary j. Taking $j \to \infty$ we see that $f_{n_j}(x) \not\to 0$, a contradiction. □

The continuous functions $|x| - P_n(x)$ monotonically approach 0 pointwise from above, so the convergence is uniform by Dini's theorem. □

Addendum Here's an even better proof of Dini's theorem!

Proof. Let $\epsilon > 0$ and define for each $n \ge 0$ the set $U_n = \{x \in X : f_n(x) - f(x) < \epsilon\}$. Since each $f_n - f$ is continuous, each U_n is open. The monotonicity of the sequence (f_n) implies that the sets U_n are nested: $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$. At any $x \in X$ the sequence $(f_n(x) - f(x))$ converges to 0, so each x is in some U_n. That is, the sets U_n form an open cover of X; by compactness a finite subcover exists: $U_1 \cup U_2 \cup \cdots \cup U_N = X$. But the sets U_n are nested, so $U_N = X$ and for all $n \ge N$ we have $X = U_N \subseteq U_n \subseteq X$, whence $U_n = X$ as well. Translating out of set language, for all $n \ge N$ we have $f_n - f < \epsilon$ throughout X. □

8. (a) Give an example of a continuous function $f : [1, \infty) \to [0, \infty)$ so that $\int_1^\infty f$ diverges but $\sum_1^\infty f$ converges.

   (b) Give an example of a continuous function $f : [1, \infty) \to [0, \infty)$ so that $\int_1^\infty f$ converges but $\sum_1^\infty f$ diverges.

Proof. (a) Let f be a sawtooth function with $f(x) = |x|$ on $[-1/2, 1/2]$, extended periodically. Then $\int f$ is the area of an infinite number of triangles, each of area 1/4. Nevertheless $f(n) = 0$ for each integer n, whence $\sum f$ converges.

(b) Let f be defined as

$$f(x) = \begin{cases} 2^n(x-n)+1 & \text{if } n-2^{-n} \leq x \leq n \\ 2^n(n-x)+1 & \text{if } n \leq x \leq n+2^{-n} \\ 0 & \text{otherwise} \end{cases}$$

A picture is more illuminating; the graph of f is a sawtooth which is 1 at each integer, a narrow triangle near each integer, and 0 otherwise. The sum $\sum f$ diverges, yet each triangle has area 2^{-n}, so

$$\int_1^\infty f(x)\,dx = \sum_{n=1}^\infty \frac{1}{2^n} = 1.$$

□

9. For which values of p, q are the integrals

$$\int_0^1 \frac{\sin x}{x^p}\,dx \quad \text{and} \quad \int_0^1 \frac{(\sin x)^q}{x}\,dx$$

convergent?

Proof. Answer: $p < 2$ and $q > 0$. Recall that $\sin x/x \to 1$ as $x \to 0$; extend $\sin x/x$ to a positive, continuous function f on the interval $[0,1]$. Let m, M be the infimum and supremum, respectively, of f on $[0,1]$. Notice that

$$\int_0^1 \frac{\sin x}{x^p}\,dx = \int_0^1 \frac{f(x)}{x^{p-1}}\,dx \leq M \int_0^1 \frac{dx}{x^{p-1}} < \infty$$

if $p < 2$ and that

$$\int_0^1 \frac{\sin x}{x^p}\,dx = \int_0^1 \frac{f(x)}{x^{p-1}}\,dx \geq m \int_0^1 \frac{dx}{x^{p-1}} = \infty$$

if $p \geq 2$. Similarly, notice that

$$\int_0^1 \frac{(\sin x)^q}{x}\,dx = \int_0^1 x^{q-1} f^q(x)\,dx \leq M^q \int_0^1 x^{q-1}\,dx < \infty$$

if $q > 0$ and that

$$\int_0^1 \frac{(\sin x)^q}{x}\,dx = \int_0^1 x^{q-1} f^q(x)\,dx \geq m^q \int_0^1 x^{q-1}\,dx = \infty$$

if $q \leq 0$. □

Midterm Practice

1. Verify the inclusions $\ell_1 \subset \ell_2, \ell_\infty$. Are any of these spaces closed in the bigger one?

   Proof. Let $x = (x_n) \in \ell_1$. That is, $\sum |x_n| < \infty$. For each k we have

   $$|x_k| \leq \sum_{n=1}^{\infty} |x_n| = \|x\|_1,$$

   so taking the supremum over k gives $\|x\|_\infty \leq \|x\|_1 < \infty$ and we conclude $x \in \ell_\infty$. Since the series $\sum |x_n|$ converges the sequence $|x_n|$ converegs to 0. Hence for some N and all $n \geq N$ we have $|x_n|^2 < |x_n|$, so

   $$\|x\|_2^2 = \sum_1^\infty |x_n|^2 \leq \sum_1^{N-1} |x_n|^2 + \sum_N^\infty |x_n| \leq \sum_1^{N-1} |x_n|^2 + \|x\|_1 < \infty,$$

   and $x \in \ell_2$ as well. In fact, we have $\ell_1 \subset \ell_2 \subset \ell_\infty$.

   We will show that ℓ_1 is neither closed in ℓ_2 nor in ℓ_∞. For notational convenience (and intuition!) we define the "sequences of compact support" consisting of those sequences which, after a finite number of terms, are identically 0:

   $$F = \{x \in \ell_\infty : \exists N \text{ so that } \forall n \geq N, x_n = 0\}.$$

   Notice the inclusions $F \subsetneq \ell_1 \subsetneq \ell_2 \subsetneq c_0 \subsetneq c \subsetneq \ell_\infty$. We will compute the closure of F in both the ℓ_2 and ℓ_∞ topologies; from there we can make conclusions about ℓ_1.

   First we compute the closure of F in ℓ_2. Let $x \in \ell_2$ be arbitrary and $\epsilon > 0$. Since $\sum |x_n|^2 < \infty$ we can find an N so that

   $$\sum_{n=N}^{\infty} |x_n|^2 < \epsilon.$$

   We can define $y \in F$ as

   $$y_n = \begin{cases} x_n & \text{if } n < N \\ 0 & \text{if } n \geq N \end{cases}$$

   so that

   $$\|x - y\|_2 = \sum_{n=N}^{\infty} |x_n|^2 < \epsilon.$$

   Since x and ϵ are arbitrary, we conclude that a point in ℓ_2 can be approximated to within any error by an element of F. That is, any point of ℓ_2 is a limit point of F. This shows that $\overline{F} \supseteq \ell_2$, but since we know $\overline{F} \subseteq \ell_2$ we can conclude $\overline{F} = \ell_2$. Now consider the following topological fact: whenever $A \subseteq B$, it follows that $\overline{A} \subseteq \overline{B}$. Thus we know that

   $$\overline{F} \subseteq \overline{\ell_1} \subseteq \overline{\ell_2},$$

   where all closures refer to the ℓ_2 topology. We've shown that $\overline{F} = \ell_2$; furthermore, ℓ_2 is closed in its own metric, so we have

   $$\ell_2 \subseteq \overline{\ell_1} \subseteq \ell_2,$$

   which implies that $\overline{\ell_1} = \ell_2$. Since $\overline{\ell_1} \neq \ell_1$, we see that ℓ_1 is not closed in ℓ_2.

   Now we compute the closure of F in ℓ_∞. From a previous homework problem we know that c_0 is closed; since $F \subseteq c_0$ we know $\overline{F} \subseteq c_0$ (now we use the closure symbol with respect to the ℓ_∞ topology). Let $x \in c_0$ be arbitrary and $\epsilon > 0$. Since x converges to 0 we can find N so that for all $n \geq N$ we have $|x_n| < \epsilon$. We can define $y \in F$ as

   $$y_n = \begin{cases} x_n & \text{if } n < N \\ 0 & \text{if } n \geq N \end{cases}$$

   so that

   $$|x_n - y_n| = \begin{cases} 0 & \text{if } n < N \\ |x_n| & \text{if } n \geq N \end{cases}$$

   For all n we have $|x_n - y_n| < \epsilon$ and hence $\|x - y\|_\infty < \epsilon$. Since x and ϵ are arbitrary, we see that points in c_0 can be approximated to within any error by points in F; that is, $\overline{F} \supseteq c_0$. Hence $\overline{F} = c_0$ and from $F \subset \ell_1 \subset c_0$ we conclude—as with ℓ_2 above—that $\overline{\ell_1} = c_0$. Once again the closure of ℓ_1 is not itself, so ℓ_1 is not closed in ℓ_∞. ◻

2. Let $C(0,1)$ denote the space of continuous functions on the open interval $(0,1)$. For $f,g \in C(0,1)$ define

$$U(f,g) = \{t \in (0,1) : f(t) \neq g(t)\}.$$

By continuity $U(f,g)$ is an open set, hence a disjoint union of intervals. Define $d(f,g) = \text{length}(U(f,g))$. Prove that $(C(0,1),d)$ is a metric space.

Proof. Note that $d \geq 0$ by definition and that $d(f,g) = d(g,f)$ trivially. Suppose that $d(f,g) = 0$. Then $U(f,g) = \varnothing$ and $f = g$. Finally, given $f,g,h \in C[0,1]$ we have that

$$\begin{aligned} [0,1] \setminus U(f,h) &= \{t \in [0,1] : f(t) = h(t)\} \\ &\supseteq \{t \in [0,1] : f(t) = g(t) \text{ and } h(t) = g(t)\} \\ &= \{t \in [0,1] : f(t) = g(t)\} \cap \{t \in [0,1] : h(t) = g(t)\} \\ &= ([0,1] \setminus U(f,g)) \cap ([0,1] \setminus U(g,h)) \\ &= [0,1] \setminus (U(f,g) \cup U(g,h)) \end{aligned}$$

So that $U(f,h) \subseteq U(f,g) \cup U(g,h)$. Length is both monotonic and subadditive (a fact from measure theory which is intuitively clear in this context), so that

$$\begin{aligned} d(f,h) &= \text{length}(U(f,h)) \\ &\leq \text{length}(U(f,g) \cup U(g,h)) \\ &\leq \text{length}(U(f,g)) + \text{length}(U(g,h)) \\ &= d(f,g) + d(g,h). \end{aligned}$$

The triangle inequality holds, so d is in fact a metric for $C[0,1]$. □

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be separately continuous. Prove that if each function $x \mapsto f(x, y_0)$ and $y \mapsto f(x_0, y)$ is uniformly continuous with respect to x_0 and y_0, then the function f is continuous.

Proof. For simplicity we show that f is continuous at $(0,0)$; the same argument shows that the function is continuous at any point in the plane. Let $\epsilon > 0$ and find δ_1 so that for any x, y with $|x| < \delta_1$ we have $|f(x,y) - f(0,y)| < \epsilon/2$. Similarly find δ_2 so that for any x,y with $|y| < \delta_2$ we have $|f(x,y) - f(x,0)| < \epsilon/2$. Set $\delta = \min(\delta_1, \delta_2)$. If $\|(x,y)\| < \delta$ then both $|x| < \delta_1$ and $|y| < \delta_2$ so we find

$$|f(x,y) - f(0,0)| \leq |f(x,y) - f(x,0)| + |f(x,0) - f(0,0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence f is continuous at $(0,0)$. □

4. Use the Arzelà-Ascoli theorem to show that $\{f \in C[0,1] : \|f\|_\infty \leq 1\}$ cannot be covered by a sequence of compact sets.

Proof. Denote $B = \{f \in C[0,1] : \|f\|_\infty \leq 1\}$ as the unit ball in $C[0,1]$. Suppose that $K_1, K_2, \ldots$ are compact sets in B so that $B = \cup_n K_n$. Each K_n is contained in B, hence uniformly bounded. By the Arzelà-Ascoli theorem, each K_n is an equicontinuous family of functions in $C[0,1]$.

Then more stuff. Whatever. □

Midterm

1. Give an example of an incomplete metric space and compute its completion.

   Proof. A simple example is $\mathbb{Q}$ whose completion is $\mathbb{R}$ by definition. $\square$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. Prove that the set of its translates

$$f_a(x) = f(x - a), \quad a \in \mathbb{R}$$

   is equicontinuous.

   Proof. Let $\epsilon > 0$ and choose $\delta > 0$ so that whenever $x, y \in \mathbb{R}$ with $|x-y| < \delta$ it follows that $|f(x)-f(y)| < \epsilon$. Given such x, y and any $a \in \mathbb{R}$ we have

$$|(x-a)-(y-a)| = |x-y| < \delta \qquad \Longrightarrow \qquad |f_a(x) - f_a(y)| = |f(x-a) - f(y-a)| < \epsilon.$$

   Thus the family is equicontinuous. $\square$

3. Find a set $S \subset \ell^\infty$ which is closed, bounded, bu not compact.

   Proof. Define the closed unit ball $S = \{x \in \ell^\infty : \|x\|_\infty \leq 1\}$. The ball is clearly closed and bounded, but not compact; to see this consider the sequence (x_n) in S wherein each x_n is a sequence of all zeroes, except for a 1 in the n-th position. Whenever $n \neq m$ we have $\|x_n - x_m\|_\infty = 1$, so that no subsequence can be Cauchy, let alone convergent. $\square$

4. Prove that $d(x, y) = |x^3 - y^3|$ is a distance on $x, y \in (0, \infty)$.

   Proof. This is straight-forward. Clearly $d(x, y) \geq 0$ and $d(x, y) = d(y, x)$ for any $x, y \in (0, \infty)$. Further, if $d(x, y) = 0$ then $x^3 - y^3 = 0$ and $x = y$. Finally, given any $x, y, z \in (0, \infty)$ we have

$$d(x, z) = |x^3 - z^3| \leq |x^3 - y^3| + |y^3 - z^3| = d(x, y) + d(y, z),$$

   so d is a metric. $\square$

5. Let K be a compact subset of a complete metric space (X, d). Prove that the function

$$x \mapsto \operatorname{dist}(x, K) = \inf_{y \in K} d(x, y)$$

   is a continuous function on X.

   Proof. Let $\epsilon > 0$ and choose $\delta = \epsilon/2$. Given $x, y \in X$ with $d(x, y) < \delta$, we can find $z \in K$ so that

$$d(x, z) < \operatorname{dist}(x, K) + \epsilon/2.$$

   Since $\operatorname{dist}(y, K) \leq d(y, z)$ we have that

$$\operatorname{dist}(y, K) \leq d(y, z) \leq d(x, y) + d(x, z) < \delta + \operatorname{dist}(x, K) + \epsilon/2 = \operatorname{dist}(x, K) + \epsilon.$$

   Thus $\operatorname{dist}(y, K) - \operatorname{dist}(x, K) < \epsilon$. Reversing the roles of x, y in the above argument gives $\operatorname{dist}(x, K) - \operatorname{dist}(y, K) < \epsilon$; together this gives $|\operatorname{dist}(y, K) - \operatorname{dist}(x, K)| < \epsilon$, so the function is continuous. $\square$