Math 5C Extra Credit Problems

A solution to any of these problems is worth 3 perfect quiz scores. Present a complete solution to me in office hours to receive credit. Each of these problems is very difficult and will require perseverance and cleverness to solve. On that note, use absolutely any resource you want in figuring out a problem; credit comes from demonstrating understanding to me. Each problem has a solution using ideas from this course, although other methods might exist.

Oh, and these problems are pretty fun.

1. For positive numbers \( t \), let \( \{ t \} \) denote the fractional (or decimal) part of \( t \). For example, \( \{3\} = 0 \) and \( \{7.5\} = 0.5 \). Evaluate

\[
\int_0^1 \left\{ \frac{1}{x} \right\}^2 \, dx.
\]

2. Define the Riemann zeta function as \( \zeta(k) = \sum_{n=1}^{\infty} n^{-k} \). Evaluate \( \sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k} \).

3. From one version of the second midterm, we know that \( \sum_{n=0}^{\infty} \left( \frac{2n}{n} \right)^{-1} \) converges. What is the exact value of the sum?

4. Let \( R \) be a bounded region in \( \mathbb{R}^2 \) with smooth boundary and assume the origin is inside \( R \). Let \( f \) be a scalar function defined so that \( f = 0 \) on \( \partial R \) and for every smooth function \( g \),

\[
\iint_R g \Delta f \, dA = g(0).
\]

In other words, \( f \) is the Green function for \( R \) with singularity at the origin. Now suppose that the region \( R \) moves in time. At a point on the boundary, the normal velocity \( V \) of \( \partial R \) is given by

\[
V = \frac{\partial f}{\partial n}.
\]

As \( R \) changes in time, \( f \) also changes to preserve the properties in its definition. Given a harmonic function \( g \), prove that

\[
\frac{d}{dt} \iint_R g \, dA = g(0).
\]

This fact has big implications in mathematical physics.

5. Let \( R \) be a nice region in \( \mathbb{C} \). Learn about Stokes’ theorem in this case:

\[
\iint_R \frac{\partial f}{\partial \bar{z}} \, d\bar{z} \wedge dz = \oint_{\partial R} f \, dz.
\]

Use this formula to prove the Cauchy integral theorem: if \( f : \mathbb{C} \to \mathbb{C} \) is analytic, \( \int_{\partial R} f \, dz = 0 \).