Math 5C Spring 2010
Exam 2 Solutions
Version B

May 7, 2010

M. Choice	
F. Resp. 1	
F. Resp. 2	
F. Resp. 3	
F. Resp. 4	
Total	

Name______________________________

Perm No.______________________________

Directions:

1. There are 125 points on this exam; 100 points = 100%.
2. Each multiple choice problem is 5 points.
3. Each multiple choice problem has exactly one best answer.
4. No multiple choice problem requires heavy computation.
5. Each free response problem is 20 points.
6. Free response questions require justification; no work, no credit.
7. **A blank free-response problem is awarded 5 points.**
8. No notes, books, or electronic devices are allowed.

Multiple Choice

1. (C) They should add to 0.
2. (E) $e^{-x}/(1-e^{-x}) = 1/(e^x - 1)$
3. (E) ∇f must have zero curl and $f\nabla f$ must also due to the product rule. C and D then are both the zero vector field.
4. (B) Taking $N \to \infty$ gives the answer
5. (C) Want zero curl
6. (B) The comparison test says $\sum a_n$ converges when $\sum b_n$ does. Thus $\sum(-1)^n a_n$ converges absolutely.
7. (C) Use Green's theorem
8. (E) A diverges by root test, B then doesn't converge absolutely, C can't converge either, and $\sum \sqrt[n]{a_n}$ can't converge because the terms don't go to 0.
9. (C) If a statement is true, then all of the ones above it must be true also. If statement 51 were true, we'd have a contradiction, so statements 51 through 100 must be false. Since statement 51 is false, there must be at least 50 true statements. The only possibility is that statements 1 through 50 are true, the rest false.

Free Response

1. The product rule gives

$$\nabla \cdot (f\nabla g - g\nabla f) = \nabla f \cdot \nabla g + f\nabla \cdot \nabla g - \nabla g \cdot \nabla f - g\nabla \cdot \nabla f = f\Delta g - g\Delta f.$$

From the divergence theorem,

$$\iint_{\partial B} (f\nabla g - g\nabla f) \cdot d\mathbf{A} = \iiint_B (f\Delta g - g\Delta f)\, dV.$$

2. Notice that ∂S is oriented counterclockwise.

$$\oint_{\partial S} z^2\mathbf{G} \cdot d\mathbf{r} = \int_0^{2\pi} 1^2(-\sin t, \cos t, \sin t \cos t e^{\cos^2 t}) \cdot (-\sin t, \cos t, 0)\, dt = 2\pi.$$

We also can write

$$\iint_S \nabla z^2 \times \mathbf{G} \cdot d\mathbf{A} = \iint_S 2z(0,0,1) \times \mathbf{G} \cdot d\mathbf{A} = \iint_S 2z(-x,-y,0) \cdot d\mathbf{A}.$$

Parametrize the cone as $\mathbf{r}(u,v) = (u\cos v, u\sin v, u)$, where $0 \le u \le 1$ and $0 \le v \le 2\pi$.

$$\iint_S \nabla z^2 \times \mathbf{G} \cdot d\mathbf{A} = \int_0^{2\pi} \int_0^1 2u(-u\cos v, -u\sin v, 0) \cdot (-u\cos v, -u\sin v, u)\, du\, dv = \frac{\pi}{2}$$

Finally, integration by parts gives

$$\iint_S z^2(\nabla \times \mathbf{G}) \cdot d\mathbf{A} = \oint_{\partial S} z^2\mathbf{G} \cdot d\mathbf{r} - \iint_S \nabla z^2 \times \mathbf{G} \cdot d\mathbf{A} = 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}.$$

3. Recall that

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{36}+\cdots=\frac{1}{(2\cdot 1)^2}+\frac{1}{(2\cdot 2)^2}+\frac{1}{(2\cdot 3)^2}+\cdots=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{24}.$$

We can write

$$\begin{aligned}
S &= 1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\cdots\\
\frac{\pi^2}{24} &= \quad\ \ \frac{1}{4}+\qquad\ \frac{1}{16}+\qquad\ \frac{1}{36}+\cdots\\
\frac{\pi^2}{24} &= \quad\ \ \frac{1}{4}+\qquad\ \frac{1}{16}+\qquad\ \frac{1}{36}+\cdots
\end{aligned}$$

and add the equations to get

$$S+\frac{\pi^2}{24}+\frac{\pi^2}{24}=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}.$$

Thus $S=\pi^2/12$.

4. Define $a_n=\binom{2n}{n}^{-1}=(n!)^2/(2n)!$. Then

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)!^2}{(2n+2)!}\cdot\frac{(2n)!}{n!^2}=\lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n+2)}=\frac{1}{4}<1.$$

By the ratio test, $\sum a_n$ converges.