Math 5C Spring 2010
Exam 2 Solutions
Version A

May 7, 2010

Name________________________________

Perm No.________________________________

M. Choice	
F. Resp. 1	
F. Resp. 2	
F. Resp. 3	
F. Resp. 4	
Total	

Directions:

1. There are 125 points on this exam; 100 points = 100%.
2. Each multiple choice problem is 5 points.
3. Each multiple choice problem has exactly one best answer.
4. No multiple choice problem requires heavy computation.
5. Each free response problem is 20 points.
6. Free response questions require justification; no work, no credit.
7. **A blank free-response problem is awarded 5 points.**
8. No notes, books, or electronic devices are allowed.

Multiple Choice

1. (C) We want $\partial_y(ax \ln y) = \partial_x(2y + x^2/y)$.
2. (D) The harmonic series
3. (C) Use the divergence theorem.
4. (A) Use $\nabla \cdot \nabla \times \mathbf{F} = 0$ and the divergence theorem
5. (B)
6. (B)
7. (D) $1/(1 - \cos^2\theta) = 1/\sin^2\theta = \csc^2\theta$
8. (D) One sixteenth of the given sum
9. (E) Call the honest guy Steve. Given any other politician (say, Joe), one among Steve and Joe must be crooked. It's not Steve, so it's Joe. This is true for every other politician, so 99 of them are crooked.

Free Response

1. For simplicity, call the vector field $\mathbf{F}$. Let B be the surface $x^2 + y^2 \leq 1$, $z = 0$, oriented downward. Together with S, they enclose the solid hemisphere R. Note that
$$\iiint_R \nabla \cdot \mathbf{F}\, dV = \iiint_R dV = \frac{2\pi}{3}.$$
On B, the unit normal is $(0, 0, -1)$. Thus
$$\iint_B \mathbf{F} \cdot d\mathbf{A} = -\iint_B (x^2 + y^2)\, d\sigma = -\int_0^{2\pi} \int_0^1 r^3\, dr\, d\theta = -\frac{\pi}{2}.$$
Finally, by the divergence theorem,
$$\iint_R \mathbf{F} \cdot d\mathbf{A} = \iiint_R \nabla \cdot \mathbf{F}\, dV - \iint_B \mathbf{F} \cdot d\mathbf{A} = \frac{2\pi}{3} - \frac{-\pi}{2} = \frac{7\pi}{6}.$$
2. From the product rule for gradients,
$$f\nabla g + g\nabla f = \nabla(fg).$$
The integral of a gradient on a closed curve is zero:
$$\oint_{\partial S} (f\nabla g + g\nabla f) \cdot d\mathbf{r} = \oint_{\partial S} \nabla(fg) \cdot d\mathbf{r} = 0.$$
Alternatively, Stokes' theorem could be used.
3. Set $a_n = (1 - 1/n)^{n^2}$. Then
$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e} < 1,$$
so the series converges by the root test.

4. Rewrite $F_n = F_{n+2} - F_{n+1}$. By definition of an infinite series,

$$\sum_{n=0}^{\infty} \frac{F_n}{F_{n+2}F_{n+1}} = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{F_{n+2} - F_{n+1}}{F_{n+2}F_{n+1}} = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{F_{n+1}} - \frac{1}{F_{n+2}}.$$

This sum telescopes:

$$\sum_{n=0}^{\infty} \frac{F_n}{F_{n+2}F_{n+1}} = \lim_{N\to\infty} \frac{1}{F_1} - \frac{1}{F_{N+2}} = \frac{1}{F_1} = 1.$$