Math 5C Spring 2010
Exam 3 Solutions

May 28, 2010

M. Choice	
F. Resp. 1	
F. Resp. 2	
F. Resp. 3	
F. Resp. 4	
Total	

Name________________________________

Perm No.______________________________

Directions:

1. There are 125 points on this exam; 100 points = 100%.
2. Each multiple choice problem is 5 points.
3. Each multiple choice problem has exactly one best answer.
4. No multiple choice problem requires heavy computation.
5. Each free response problem is 20 points.
6. Free response questions require justification; no work, no credit.
7. **A blank free-response problem is awarded 5 points.**
8. No notes, books, or electronic devices are allowed.

Multiple Choice

1. (E)
2. (B) Since all derivatives of cosine are bounded by 1, $R_n(0.1) \leq (0.1)^3/3! = 1/6000$
3. (B) The average of the left and right hand limits of f at 0
4. (E) The answer is $\sqrt{2\pi}$
5. (D) The interval of convergence must be centered at 0
6. (C) 7! times the coefficient of x^7
7. (B) Since 3 is within the interval of convergence, 2 is also
8. (B) The uniqueness principle of harmonic functions
9. (B) Each sign can have at most one true statement. If one sign has all false statements, then the other sign has exactly two false statements. But then the first sign had a true statement, which is a contradiction. Thus each sign has one true statement: "Exactly two of the other sign's statements are false."

Free Response

1. For small t, $\ln(1+t) = t - t^2/2 + O(t^3)$. Thus
$$\ln\left(1+\frac{1}{n}\right)^n = n\ln\left(1+\frac{1}{n}\right) = 1 - \frac{1}{2n} + O(n^{-2})$$
Undoing the logarithm,
$$\left(1+\frac{1}{n}\right)^n = \exp\left(1 - \frac{1}{2n} + O(n^{-2})\right) = e\cdot\exp\left(-\frac{1}{2n} + O(n^{-2})\right)$$
For small t, $\exp(t) = 1 + t + O(t^2)$, so
$$\left(1+\frac{1}{n}\right)^n = e\cdot\left[1 + \left(-\frac{1}{2n} + O(n^{-2})\right) + O(n^{-2})\right] = e - \frac{e}{2n} + O(n^{-2})$$
The limit is
$$\lim_{n\to\infty} n\left(\left(1+\frac{1}{n}\right)^n - e\right) = \lim_{n\to\infty} n\left(-\frac{e}{2n} + O(n^{-2})\right) = \lim_{n\to\infty} -\frac{e}{2} + O(n^{-1}) = -\frac{e}{2}$$

2. Multiply the given formula by x.
$$\frac{x^2}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}x^{n+1}.$$
Differentiate:
$$\frac{2x(e^x-1) - x^2e^x}{(e^x-1)^2} = \sum_{n=0}^{\infty} \frac{(n+1)B_n}{n!}x^n.$$
Set $x = 1$ to get an answer of $(e-2)/(e-1)^2$.

3. For integers $n \geq 1$ we have
$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{\pi^2}{3} \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx = \frac{4(-1)^n}{n^2} \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx = 0. \end{aligned}$$
The Fourier series is
$$a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) = \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right)$$

4. Since $\Delta f = 0$, the mean-value property gives
$$\iint_{\partial B} f\,d\sigma = \text{area}(\partial B)f(1,2,2) = 4\pi\ln 8.$$