# Math 5C Spring 2010
# Final Exam
# Version A

**June 9, 2010**

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|---|---|
| M. Choice | |
| F. Resp. 1 | |
| F. Resp. 2 | |
| F. Resp. 3 | |
| F. Resp. 4 | |
| F. Resp. 5 | |
| F. Resp. 6 | |
| F. Resp. 7 | |
| F. Resp. 8 | |
| F. Resp. 9 | |
| F. Resp. 10 | |
| F. Resp. 11 | |
| F. Resp. 12 | |
| Total | |

**Name**______________________________

**Perm No.**___________________________

Directions:

1. There are 340 points on this exam; 250 points = 100%.
2. Each multiple choice problem is 5 points.
3. Each multiple choice problem has exactly one best answer.
4. No multiple choice problem requires heavy computation.
5. Each free response problem is 20 points.
6. Free response questions require justification; no work, no credit.
7. **A blank free-response problem is awarded 5 points.**
8. You may use any hand-written notes.
9. No books or electronic devices are allowed.

**Multiple Choice**

1. Let $S$ be the sphere of radius $R$ centered at the origin in $\mathbb{R}^3$. Evaluate $\iint_S y^2\,d\sigma$.
   (a) $4\pi R^2$
   (b) $4\pi R^4$
   (c) $4\pi R^2/3$
   (d) $4\pi R^4/3$
   (e) None of the above

2. Suppose $(a_n)$ is a sequence so that for each positive integer $N$,
$$\sum_{n=1}^{N} a_n = 2 - \frac{1}{N}.$$
   What is $\lim_{n\to\infty} a_n$?
   (a) 0
   (b) 1
   (c) 2
   (d) $\infty$
   (e) None of the above

3. Using the inner product for functions on $[-\pi, \pi]$, if $\|f\| = 0$ what must be true?
   (a) $f$ is identically zero
   (b) $f$ is not in $L^2$
   (c) $f$ is either a sine or cosine
   (d) $f$ is discontinuous
   (e) $f$ has a singularity

4. Given a region $R$ in $\mathbb{R}^3$, what is $\partial R$?
   (a) The derivative of $R$
   (b) The volume of $R$
   (c) The integral of $R$
   (d) The boundary of $R$
   (e) The set of all subsets of $R$

5. If $\sin^2 x$ is expanded in a Fourier series, what is the coefficient of $\sin x$?
   (a) 0
   (b) 1
   (c) $\sin x$
   (d) $1/2$
   (e) None of the above

6. Suppose $f$ is a function so that $f(x) = \sum a_n x^n$ in an open neighborhood of zero. By definition, what do we call $f$?
   (a) Smooth
   (b) Analytic
   (c) Tayloriffic
   (d) Convergent
   (e) Harmonic

7. Let $D$ be the unit disk centered at $(1, 0)$. Evaluate $\iint_D (2y + 3)\, dA$.
   (a) $2\pi$
   (b) $3\pi$
   (c) $4\pi$
   (d) $5\pi$
   (e) None of the above

8. What is the radius of convergence for $\sum(-1)^n x^n/n^2$?
   (a) 0
   (b) 1
   (c) $\pi^2/6$
   (d) $\infty$
   (e) None of the above

9. For $x$ near 0, $O(x^2) + O(x^4) =$
   (a) $O(x^2)$
   (b) $O(x^3)$
   (c) $O(x^4)$
   (d) $O(x^6)$
   (e) None of the above

10. For a square in $\mathbb{R}^2$, how many linearly-independent solutions of Laplace's equation exist?
    (a) 0
    (b) 1
    (c) 2
    (d) 1,453,998
    (e) Infinitely many

11. Sum the double series

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m^2n^2}$$

(a) $\pi^4/36$
(b) $\pi^4/90$
(c) $\pi^2/6$
(d) $\pi^2/3$
(e) None of the above

12. Among the following, whose definition of the integral is the most general?

(a) Newton
(b) Leibniz
(c) Riemann
(d) Darboux
(e) Lebesgue

13. Suppose that $f(x)$ is a $2\pi$-periodic function on $\mathbb{R}$ and that $f(x) = e^{-x}$ for $-\pi \le x \le \pi$. If $g(x)$ is the Fourier series of $f$, what is $g(\pi)$?

(a) $e^{\pi}$
(b) $e^{-\pi}$
(c) $\sinh \pi$
(d) $\cosh \pi$
(e) None of the above

14. The magnetostatic field $\mathbf{B}$ and the current density $\mathbf{J}$ are related by $\mathbf{J} = \nabla \times \mathbf{B}$. Given an oriented surface $S$ with boundary, the flux of $\mathbf{J}$ through $S$ is the line integral of $\mathbf{B}$ along the boundary of $S$. This follows from which theorem?

(a) Green's theorem
(b) Divergence theorem
(c) Stokes' theorem
(d) Fubini's theorem
(e) None of the above

15. If we use

$$\pi = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots,$$

how many terms do we need to approximate 3 digits (after the decimal point) of $\pi$ according to the alternating series test?

(a) Around 80
(b) Around 800
(c) Around 8,000
(d) Around 80,000
(e) Around 800,000

16. Solve the following equation for $\delta$.

$$10 + 2\delta + 2\delta^2 + 2\delta^3 + \cdots = 5 + 5\delta + 5\delta^2 + 5\delta^3 + \cdots$$

(a) 7/10
(b) 10/13
(c) 5/3
(d) No solution
(e) None of the above

17. Which of these gives the area of a region $R$ in $\mathbb{R}^2$?

(a) $\oint_{\partial R} y\,dx$
(b) $\oint_{\partial R} x\,dy$
(c) $\oint_{\partial R} -y\,dx + x\,dy$
(d) $\oint_{\partial R} y\,dx - x\,dy$
(e) $\oint_{\partial R} x\,dx - y\,dy$

18. Which of these functions is harmonic in $\mathbb{R}^2$?

(a) $x^2 + y^2$
(b) $x^3 - y^3$
(c) $\ln(x^4 + y^4)$
(d) $e^x \cos y$
(e) $\cos x \cos y$

19. (Bonus Problem 1) What is the answer to the next question?

(a) C
(b) D
(c) E
(d) None of the above
(e) All of the above

20. (Bonus Problem 2) What is the answer to the previous question?

(a) C
(b) D
(c) E
(d) None of the above
(e) B

## Free Response

1. Let $C$ be the curve in $\mathbb{R}^2$ given by $y = \ln \sec x$, $0 \le x \le \pi/4$. Evaluate $\displaystyle\int_C \sec x \, ds$.

2. Let $R$ be the unit disk in $\mathbb{R}^2$ centered at $(1,0)$. Evaluate $\displaystyle\iint_R \sqrt{4-x^2-y^2}\,dA$.

3. Let $R$ be the region in the <u>first octant</u> of $\mathbb{R}^3$ in which $1 \le x^2 + y^2 + z^2 \le 4$. Evaluate

$$\iiint_R xyz \, dV$$

4. Let $B$ be a ball in $\mathbb{R}^3$ and $f$ be a harmonic function. Suppose that $g$ is a smooth function (not necessarily harmonic) so that $f = g$ on $\partial B$. Show that

$$\iiint_B \nabla f \cdot \nabla g \, dV = \iiint_B \|\nabla f\|^2 \, dV$$

Suggestion: Consider the outward flux of $f \nabla f$.

5. Let $S$ be the upwardly-oriented cone in $\mathbb{R}^3$ given by $z = 1 - \sqrt{x^2+y^2}$, $z \geq 0$. Define

$$f(x,y,z) = e^{z+\sqrt{x^2+y^2}} \qquad \text{and} \qquad \mathbf{G}(x,y,z) = (z, -z, z\ln(1+z^2))$$

Evaluate

$$\iint_S \nabla f \times \mathbf{G} \cdot d\mathbf{A}$$

Hint: Note that $f$ and $\nabla \times \mathbf{G}$ are much simpler than $\nabla f$ and $\mathbf{G}$ on the given surface.

6. Evaluate

$$\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$$

Suggestion: Try telescoping.

7. Determine—with justification—all real numbers $p$ so that the following series converges.

$$\sum \frac{1}{n^p}\left(1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}\right)$$

Suggestion: Use comparison for $p \leq 1$ and limit comparison for $p > 1$.

8. The Bell numbers $B_n$ count the number of ways to partition $n$ objects into any number of unordered sets. Surprisingly, they satisfy

$$e^{e^x} = e \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

Use this to evaluate

$$\sum_{n=0}^{\infty} \frac{n+1}{n!} B_n$$

(Also surprising: $B_n$ is the imaginary part of $2n!/(\pi e) \int_0^{\pi} \exp \exp \exp(i\theta) \sin(n\theta)\, d\theta$)

9. For all $-\pi < x < \pi$ the following holds:

$$\cos x - \frac{\cos 2x}{2} + \frac{\cos 3x}{3} - \frac{\cos 4x}{4} + \cdots = \ln(2\cos(x/2))$$

Use Parseval's identity to evaluate

$$\int_{-\pi}^{\pi} \ln^2(2\cos(x/2))\,dx$$

10. Let $B$ be a ball in $\mathbb{R}^3$. Recall the following theorem:

> If $h$ is harmonic and $h = 0$ on $\partial B$, then $h = 0$ inside $B$.

Using this theorem, prove the following generalization:

> If $f$ and $g$ are smooth scalar functions (<u>not</u> necessarily harmonic) such that $f = g$ on $\partial B$ and $\Delta f = \Delta g$ everywhere, then $f = g$ in $B$.

Please use complete sentences and clearly indicate where you cite the given theorem. You may also prove the result via other means.

11. Suppose that $u$ is a function on the square $0 \leq x, y \leq \pi$ so that for $0 \leq y \leq \pi$ we have $u(\pi, y) = y$ and that inside the square,

$$u(x, y) = \sum_{n=1}^{\infty} L_n \sinh(nx) \sin(ny)$$

for some constants $L_n$. Without citing the result from class, <u>derive</u> the coefficients $L_n$.

12. Using a series expansion, evaluate $\displaystyle\int_0^\infty \frac{x}{e^x - 1}\, dx$. Justify any sum/integral swaps.