Math 5C Spring 2010
Final Exam
Version B

June 9, 2010

Name______________________________

Perm No.______________________________

M. Choice	
F. Resp. 1	
F. Resp. 2	
F. Resp. 3	
F. Resp. 4	
F. Resp. 5	
F. Resp. 6	
F. Resp. 7	
F. Resp. 8	
F. Resp. 9	
F. Resp. 10	
F. Resp. 11	
F. Resp. 12	
Total	

Directions:

1. There are 340 points on this exam; 250 points = 100%.
2. Each multiple choice problem is 5 points.
3. Each multiple choice problem has exactly one best answer.
4. No multiple choice problem requires heavy computation.
5. Each free response problem is 20 points.
6. Free response questions require justification; no work, no credit.
7. **A blank free-response problem is awarded 5 points.**
8. You may use any hand-written notes.
9. No books or electronic devices are allowed.

Multiple Choice

1. Let C be the unit circle centered at the origin in $\mathbb{R}^2$. Evaluate $\displaystyle\int_C x^2\,ds$.

   (a) 0
   (b) $\pi/2$
   (c) π
   (d) 2π
   (e) None of the above

2. Suppose (a_n) is a sequence so that for each positive integer N,

   $$\sum_{n=1}^{N} a_n = 2 - \frac{1}{N}.$$

   What is $\sum_1^\infty a_n$?

   (a) 0
   (b) 1
   (c) 2
   (d) ∞
   (e) None of the above

3. Heuristically, all integration theorems are of which form?

   (a) $\displaystyle\int_R d\omega = \int_{\partial R} d\omega$
   (b) $\displaystyle\int_R d\omega = \int_{\partial R} \omega$
   (c) $\displaystyle\int_R \omega = \int_{\partial R} d\omega$
   (d) $\displaystyle\int_R \omega = \int_{\partial R} \omega$
   (e) None of the above

4. If $\sin^2 x$ is expanded in a Fourier series, what is the coefficient of $\cos 2x$?

   (a) $-1/2$
   (b) 0
   (c) 1
   (d) $1/2$
   (e) None of the above

5. A function f for which $\displaystyle\int_{-\pi}^{\pi} f(x)^2\,dx < \infty$ is said to be in which vector space?

   (a) $\mathbb{R}^2$
   (b) C^2
   (c) $\mathbb{C}^2$
   (d) L^2
   (e) H^2

6. For x near zero, $O(x) \times O(x^3) =$
   (a) $O(x)$
   (b) $O(x^2)$
   (c) $O(x^3)$
   (d) $O(x^4)$
   (e) None of the above

7. Let D be the unit disk centered at $(0, 1)$. Evaluate $\displaystyle\iint_D (2y + 3)\, dA$.
   (a) 2π
   (b) 3π
   (c) 4π
   (d) 5π
   (e) None of the above

8. Which of the following is <u>not</u> described by a harmonic function?
   (a) Electrostatic potential
   (b) Steady-state temperature
   (c) Gravitational potential
   (d) Slow, viscous fluid flow
   (e) Elasticity

9. Which of the following sets of functions is <u>not</u> orthogonal with respect to the standard inner product on $[-\pi, \pi]$?
   (a) $\{1, x, x^2, x^3, \ldots\}$
   (b) $\{\sin 2x, \sin 4x, \sin 6x, \sin 8x \ldots\}$
   (c) $\{\cos x, \cos 3x, \cos 5x, \cos 7x, \ldots\}$
   (d) $\{1, xe^{-x^2}\}$
   (e) None of the above

10. In which of the following does the monotone convergence theorem allow us to swap sum and integral?
   (a) $\displaystyle\int_0^1 \sum (-1)^n x^n / n^2 \, dx$
   (b) $\displaystyle\int_0^1 \sum \cos(\pi n) \ln x / n^3 \, dx$
   (c) $\displaystyle\int_{-1}^1 \sum x / n^2 \, dx$
   (d) $\displaystyle\int_0^\infty \sum e^{-x^2} \arctan(1/n^2) \, dx$
   (e) None of the above

11. Sum the double series

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{4^{m+n}}$$

(a) 16/9
(b) 1/9
(c) 32/9
(d) 2/9
(e) None of the above

12. Which of the following refers to solving a PDE in a bounded region given the values of the function on the boundary?

(a) Dirichlet problem
(b) Green problem
(c) Robin problem
(d) Neumann problem
(e) Whatsyo problem?

13. If 3 terms of the Taylor series are used to approximate $e^{1/10}$, what does the remainder theorem give as the error?

(a) 0
(b) 1/30000
(c) 1/80000
(d) 1/240000
(e) None of the above

14. Which of these functions is harmonic in $\mathbb{R}^2$?

(a) $x^2 + y^2$
(b) $x^3 - y^3$
(c) $\ln(x^4 + y^4)$
(d) $e^x \cosh y$
(e) $\cos x \cosh y$

15. Let f be a 2π-periodic function with $f(x) = x^2 e^x$ for $-\pi < x < \pi$. If g is the Fourier series of f, what is $g(\pi)$?

(a) $\pi^2 e^{\pi}$
(b) $\pi^2 e^{-\pi}$
(c) $\pi^2 \cosh \pi$
(d) $\pi^2 \sinh \pi$
(e) None of the above

16. What is the interval of convergence for $\sum x^n/n$?
    (a) $[-1, 1]$
    (b) $[-1, 1)$
    (c) $(-1, 1]$
    (d) $(-1, -1)$
    (e) None of the above
17. Which of these gives the area of a region R in $\mathbb{R}^2$?
    (a) $\frac{1}{2} \oint_{\partial R} y\,dx$
    (b) $\frac{1}{2} \oint_{\partial R} x\,dy$
    (c) $\frac{1}{2} \oint_{\partial R} y\,dx - x\,dy$
    (d) $\frac{1}{2} \oint_{\partial R} x\,dx - y\,dy$
    (e) $\frac{1}{2} \oint_{\partial R} -y\,dx + x\,dy$
18. Which of the following is true in an arbitrary inner product space?
    (a) $\langle u, v\rangle \geq 0$ for all vectors u and v
    (b) $\langle u, u\rangle = 0$ forces $u = 0$
    (c) $\langle u, u\rangle = \|u\|$ for all vectors u
    (d) $\langle u, v\rangle = -\langle v, u\rangle$ for all vectors u and v
    (e) There exists a finite basis
19. (Bonus Problem 1) On the island of knights and knaves, knights always tell the truth and knaves always lie. At a party of 100 people, every person says: "Some of us are knights and some of us are knaves." How many knaves are at this party?
    (a) 0
    (b) 50
    (c) 100
    (d) Can't be determined
    (e) None of the above
20. (Bonus Problem 2) On the island of knights and knaves, knights always tell the truth and knaves always lie. Four people—named A, B, C, and D— live on this island and make the statements below. Choose one person you are sure to be a knight.
    (a) D is a knave.
    (b) A is a knave.
    (c) A is a knave.
    (d) Exactly one among B and C is a knight.
    (e) (We can't conclude any of them are knights)

Free Response

1. Let C be the straight-line path in $\mathbb{R}^3$ from $(1, 1, 1)$ to $(2, 2, 2)$. Evaluate

$$\int_C \frac{(x, y, z)}{x^2 + y^2 + z^2} \cdot d\mathbf{r}$$

2. Evaluate $\displaystyle\iint_{\mathbb{R}^2} \frac{dA}{1+(x^2+y^2)^2}$.

3. Let R be the region in $\mathbb{R}^3$ inside both the cylinder $x^2 + y^2 = 1$ and sphere $x^2 + y^2 + z^2 = 4$. Evaluate

$$\iiint_R \sqrt{4 - x^2 - y^2}\, dV$$

4. Let R be a region in $\mathbb{R}^3$ with smooth boundary ∂R, oriented outward. Suppose that p is a smooth scalar function so that $p = 0$ everywhere on ∂R. Assuming that f is harmonic, show that

$$\iint_{\partial R} f \nabla p \cdot d\mathbf{A} = \iiint_R f \Delta p \, dV$$

Suggestion: Also consider the flux of $p\nabla f$.

5. Let S be the upwardly-oriented cone in $\mathbb{R}^3$ given by $z = 4 - \sqrt{x^2+y^2}$, $z \geq 0$. Define

$$f(x,y,z) = e^{z+\sqrt{x^2+y^2}} \qquad \text{and} \qquad \mathbf{G}(x,y,z) = (z, z, z\ln(1+z^2))$$

Evaluate

$$\iint_S \nabla f \times \mathbf{G} \cdot d\mathbf{A}$$

Hint: Note that f and $\nabla \times \mathbf{G}$ are much simpler than ∇f and $\mathbf{G}$ on the given surface.

6. Evaluate

$$\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$$

Suggestion: Try telescoping.

7. Determine, with justification, whether or not this series converges.

$$\sum \frac{1}{2^n}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}\right)$$

8. Evaluate

$$\sum_{n=1}^{\infty} \frac{n^2}{(n+1)!}$$

Suggestion: Consider a power series.

9. For all real x the following holds:

$$\frac{\sin x}{2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{2^3} + \frac{\sin 4x}{2^4} + \cdots = \frac{2\sin x}{5 - 4\cos x}$$

Use Parseval's identity to evaluate

$$\int_{-\pi}^{\pi} \left(\frac{2\sin x}{5 - 4\cos x} \right)^2 dx$$

10. Let B be the unit ball in $\mathbb{R}^3$ and f be a 'biharmonic' function; that is, $\Delta(\Delta f) = 0$ everywhere. Prove the following uniqueness result:

If both $f = 0$ and $\partial f/\partial n = 0$ on ∂B, then $f = 0$ everywhere inside.

Hint: The following 'Green identity' is true for every smooth function g:

$$\iiint_B (f\Delta g - g\Delta f)\, dV = \iint_{\partial B} \left(f\frac{\partial g}{\partial n} - f\frac{\partial f}{\partial n} \right) d\sigma$$

Set $g = \Delta f$ and see what happens.

11. Suppose that u is a function on the square $0 \le x, y \le \pi$ given by

$$u(x, y) = \sum_{n=1}^{\infty} L_n \sinh(nx) \sin(ny)$$

for some constants L_n. Suppose that $u(\pi, y) = \sin y \cos y$ for all $0 \le y \le \pi$; without citing the result from class, <u>derive</u> the coefficients L_n.

12. Use a series to evaluate $\int_0^1 \frac{\ln x}{1-x}\,dx$. Justify sum/integral swaps.