Math 5C Spring 2010
Final Exam
Version B Solutions

June 9, 2010

M. Choice	
F. Resp. 1	
F. Resp. 2	
F. Resp. 3	
F. Resp. 4	
F. Resp. 5	
F. Resp. 6	
F. Resp. 7	
F. Resp. 8	
F. Resp. 9	
F. Resp. 10	
F. Resp. 11	
F. Resp. 12	
Total	

Name______________________________

Perm No.___________________________

Directions:

1. There are 340 points on this exam; 250 points = 100%.
2. Each multiple choice problem is 5 points.
3. Each multiple choice problem has exactly one best answer.
4. No multiple choice problem requires heavy computation.
5. Each free response problem is 20 points.
6. Free response questions require justification; no work, no credit.
7. **A blank free-response problem is awarded 5 points.**
8. You may use any hand-written notes.
9. No books or electronic devices are allowed.

Multiple Choice

1. (C)
2. (C)
3. (B)
4. (A)
5. (D)
6. (D)
7. (D)
8. (E)
9. (A)
10. (D)
11. (B)
12. (A)
13. (C)
14. (E)
15. (C)
16. (B)
17. (E)
18. (B)
19. (C)
20. (A)

Free Response

1. Set $\mathbf{r}(t) = (t, t, t)$, $1 \le t \le 2$. Then
$$\int_C \frac{(x, y, z)}{x^2 + y^2 + z^2} \cdot d\mathbf{r} = \int_1^2 \frac{(t, t, t) \cdot (1, 1, 1)}{3t^2}\, dt = \ln 2$$

2. In polar,
$$\iint_{\mathbb{R}^2} \frac{dA}{1 + (x^2 + y^2)^2} = \int_0^{2\pi} \int_0^\infty \frac{r\, dr\, d\theta}{1 + r^4} = \frac{\pi^2}{2}$$

3. In cylindrical,
$$\iiint_R \sqrt{4 - x^2 - y^2}\, dV = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\sqrt{4 - r^2}\, dz\, dr\, d\theta = 2\pi \int_0^1 2r(4 - r^2)\, dr = 7\pi$$

4. Since $p = 0$ on ∂R, we can write
$$\iint_{\partial R} f\nabla p \cdot d\mathbf{A} = \iint_{\partial R} f\nabla p \cdot d\mathbf{A} - \iint_{\partial R} p\nabla f \cdot d\mathbf{A} = \iint_{\partial R} (f\nabla p - p\nabla f) \cdot d\mathbf{A}$$
Use the divergence theorem and the product rule to get
$$\begin{aligned}\iint_{\partial R} f\nabla p \cdot d\mathbf{A} &= \iiint_R \nabla \cdot (f\nabla p - p\nabla f)\, dV \\ &= \iiint_R (\nabla f \cdot \nabla p + f\nabla \cdot \nabla p - \nabla p \cdot \nabla f - p\nabla \cdot \nabla f)\, dV \\ &= \iiint_R (f\Delta p - p\Delta f)\, dV\end{aligned}$$
Since $\Delta f = 0$, we're done.

5. Everywhere on S, $f = e^4$, a constant. On ∂S, $\mathbf{G} = 0$. Also, $\nabla \times \mathbf{G} = (-1, 1, 0)$. The integration by parts formula becomes
$$\iint_S \nabla f \times \mathbf{G} \cdot d\mathbf{A} = \oint_{\partial S} f\mathbf{G} \cdot d\mathbf{r} - \iint_S f\nabla \times \mathbf{G} \cdot d\mathbf{A} = 0 - e^4 \iint_S \nabla \times \mathbf{G} \cdot d\mathbf{A}$$
There are four ways to proceed. We could parametrize S via $\mathbf{r}(u, v) = (u\cos v, u\sin v, 4 - u)$ and evaluate the integral. We could use the divergence theorem to say that the answer is the same as the flux through the base of the cone. We could use symmetry to say that the flux of $(-1, 0, 0)$ cancels the flux of $(0, 1, 0)$. We'll just use Stokes' theorem again:
$$\iint_S \nabla f \times \mathbf{G} \cdot d\mathbf{A} = -e^4 \int_{\partial S} \mathbf{G} \cdot d\mathbf{r} = 0$$

6. Rewrite:
$$\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1} = \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{2n - 1} - \frac{1}{2n + 1}\right) = \lim_{N\to\infty} \left(1 - \frac{1}{2N + 1}\right) = 1$$

7. Note that the given series is positive and that
$$\frac{1}{2^n}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}\right) \le \frac{1}{2^n}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) = \frac{1}{2^n}(2)$$
Since $\sum 2/2^n$ converges, the original series converges by the comparison test.

8. Start with

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots$$

Subtract off the first two terms and reindex.

$$e^x - 1 - x = \sum_{n=1}^{\infty} \frac{x^{n+1}}{(n+1)!}$$

Divide by x and take a derivative.

$$\frac{xe^x - e^x + 1}{x^2} = \frac{x(e^x - 1) - (e^x - 1 - x)}{x^2} = \frac{d}{dx}\left(\frac{e^x - 1 - x}{x}\right) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{(n+1)!}$$

Multiply by x and take one more derivative.

$$\frac{x^2 e^x - (xe^x - e^x + 1)}{x^2} = \frac{d}{dx}\left(\frac{xe^x - e^x + 1}{x}\right) = \sum_{n=1}^{\infty} \frac{n^2 x^{n-1}}{(n+1)!}$$

Set $x = 1$ to get $e - 1$ as the answer.

9. For the given function, $a_0 = 0$, $a_n = 0$, and $b_n = 1/2^n$. From Parseval's identity,

$$\int_{-\pi}^{\pi} \left(\frac{2\sin x}{5 - 4\cos x}\right)^2 dx = 2\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \pi \sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{\pi}{3}$$

10. Since $f = 0$ and $\partial f/\partial n = 0$ on ∂B, for any function g Green's identity says

$$\iiint_B (f\Delta g - g\Delta f)\, dV = 0$$

Choose $g = \Delta f$; then $\Delta g = 0$ and the equation becomes

$$\iiint_B (\Delta f)^2\, dV = 0$$

The only way the integral of a nonnegative function (e.g., $(\Delta f)^2$) can be zero is if the function is zero everywhere. So $\Delta f = 0$. Now we know f is harmonic and $f = 0$ on ∂B, so $f = 0$ inside B by the uniqueness principle of harmonic functions.

11. Set $x = \pi$ and note that $\sin y \cos y = (1/2)\sin 2y$.

$$\frac{\sin 2y}{2} = u(\pi, y) = \sum_{n=1}^{\infty} L_n \sinh(n\pi)\sin(ny)$$

Recall the orthogonality relation

$$\int_{-\pi}^{\pi} \sin nx \sin mx\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

So we integrate $u(\pi, y)\sin(ny)$ for an arbitrary integer n. If $n \neq 2$

$$0 = L_n \sinh(n\pi)\pi$$

so that $L_n = 0$ when $n \neq 2$. For $n = 2$ we get

$$\frac{\pi}{2} = L_2 \sinh(2\pi)\pi$$

so $L_2 = 1/(2\sinh 2\pi)$.

12. Expand in a geometric series.

$$\int_0^1 \frac{\ln x}{1-x}\,dx = \int_0^1 \sum_{n=0}^{\infty} x^n \ln x\,dx.$$

On $[0,1]$ each term $x^n \ln x$ is negative, so the monotone convergence theorem applies:

$$\int_0^1 \frac{\ln x}{1-x}\,dx = \sum_{n=0}^{\infty} \int_0^1 x^n \ln x\,dx.$$

Integrate by parts:

$$\int_0^1 x^n \ln x\,dx = \int_0^1 \frac{x^{n+1}\ln x}{n+1}\bigg|_0^1 - \int_0^1 \frac{x^n}{n+1}\,dx = -\frac{1}{(n+1)^2}$$

The integral is then

$$\int_0^1 \frac{\ln x}{1-x}\,dx = \sum_{n=0}^{\infty} -\frac{1}{(n+1)^2} = -\frac{\pi^2}{6}$$