Math 5C Spring 2010
Final Exam
Version A Solutions

June 9, 2010

Name______________________________

Perm No.______________________________

M. Choice	
F. Resp. 1	
F. Resp. 2	
F. Resp. 3	
F. Resp. 4	
F. Resp. 5	
F. Resp. 6	
F. Resp. 7	
F. Resp. 8	
F. Resp. 9	
F. Resp. 10	
F. Resp. 11	
F. Resp. 12	
Total	

Directions:

1. There are 340 points on this exam; 250 points = 100%.
2. Each multiple choice problem is 5 points.
3. Each multiple choice problem has exactly one best answer.
4. No multiple choice problem requires heavy computation.
5. Each free response problem is 20 points.
6. Free response questions require justification; no work, no credit.
7. **A blank free-response problem is awarded 5 points.**
8. You may use any hand-written notes.
9. No books or electronic devices are allowed.

Multiple Choice

1. (D)
2. (A)
3. (A)
4. (D)
5. (A)
6. (B)
7. (B)
8. (B)
9. (A)
10. (E)
11. (A)
12. (E)
13. (D)
14. (C)
15. (C)
16. (E)
17. (B)
18. (D)
19. (D)
20. (B)

Free Response

1. Parametrize $\mathbf{r}(t) = (t, \ln \sec t)$ with $0 \le t \le \pi/4$. Then

$$\int_C \sec x\, ds = \int_0^{\pi/4} \sec t \sqrt{1+\tan^2 t}\, dt = \int_0^{\pi/4} \sec^2 t\, dt = 1.$$

2. In polar,

$$\iint_R \sqrt{4-x^2-y^2}\, dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r\sqrt{4-r^2}\, dr\, d\theta = \frac{8}{3}\int_{-\pi/2}^{\pi/2} (1-\sin^3\theta)\, d\theta = \frac{8\pi}{3}.$$

3. In spherical,

$$\iiint_R xyz\, dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^5 \sin^3\phi \cos\phi \sin\theta \cos\theta\, d\rho\, d\phi\, d\theta = \frac{21}{16}$$

4. Using the divergence theorem and the product rule,

$$\iint_{\partial B} f\nabla f \cdot d\mathbf{A} = \iiint_B (f\nabla \cdot (\nabla f) + \nabla f \cdot \nabla f)\, dV.$$

Since f is harmonic, $\nabla \cdot (\nabla f) = 0$; also, $\nabla f \cdot \nabla f = \|\nabla f\|^2$.

$$\iint_{\partial B} f\nabla f \cdot d\mathbf{A} = \iiint_B \|\nabla f\|^2\, dV. \tag{1}$$

Since $g = f$ on ∂B, the surface integral can be rewritten:

$$\iint_{\partial B} f\nabla f \cdot d\mathbf{A} = \iint_{\partial B} g\nabla f \cdot d\mathbf{A}. \tag{2}$$

Using the divergence theorem and product rule again,

$$\iint_{\partial B} g\nabla f \cdot d\mathbf{A} = \iiint_B (g\nabla \cdot (\nabla f) + \nabla g \cdot \nabla f)\, dV = \iiint_B \nabla g \cdot \nabla f\, dV. \tag{3}$$

Combine equations (1), (2), and (3) to get the result.

5. On ∂S, $\mathbf{G} = 0$. On S, $f = e$, a constant. The integration by parts formula gives

$$\iint_S \nabla f \times \mathbf{G} \cdot d\mathbf{A} = \oint_{\partial S} f\mathbf{G} \cdot d\mathbf{r} - \iint_S f\nabla \times \mathbf{G} \cdot d\mathbf{A} = -e\iint_S \nabla \times \mathbf{G} \cdot d\mathbf{A}.$$

Use Stokes' theorem on this integral to get

$$\iint_S \nabla f \times \mathbf{G} \cdot d\mathbf{A} = -e\oint_{\partial S} \mathbf{G} \cdot d\mathbf{r} = 0.$$

6. Rewrite:

$$\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) = \lim_{N\to\infty} \sum_{n=1}^{N} \ln\left(\frac{n^2-1}{n^2}\right) = \lim_{N\to\infty} \sum_{n=1}^{N} \ln\left(\frac{n+1}{n}\right) - \ln\left(\frac{n}{n-1}\right)$$

This sum telescopes:

$$\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) = \lim_{N\to\infty} \ln\left(\frac{N+1}{N}\right) - \ln 2 = -\ln 2$$

7. For $p \leq 1$, notice that

$$1 + \frac{1}{2^p} + \cdots + \frac{1}{n^p} \geq 1$$

so that

$$\sum \frac{1}{n^p}\left(1 + \frac{1}{2^p} + \cdots + \frac{1}{n^p}\right) \geq \sum \frac{1}{n^p} = \infty,$$

so the sum diverges by the comparison test for $p \leq 1$. For $p > 1$, we try limit comparison to $1/n^p$:

$$\lim_{n\to\infty} \frac{(1/n^p)(1 + 1/2^p + 1/3^p + \cdots + 1/n^p)}{1/n^p} = \lim_{n\to\infty}\left(1 + \frac{1}{2^p} + \cdots + \frac{1}{n^p}\right) = \sum_{k=1}^{\infty} \frac{1}{k^p}$$

This series is a positive, yet finite, number since $p > 1$. The limit comparison test tells us that the given series and $\sum 1/n^p$ either both converge or diverge. For $p > 1$, the given series converges since $\sum 1/n^p$ does.

8. Multiply by x and take a derivative:

$$\frac{1}{e}\frac{d}{dx}\left(xe^{e^x}\right) = \sum_{n=0}^{\infty} \frac{(n+1)B_n}{n!}x^n$$

Setting $x = 1$ gives a sum of $e^{e-1} + e^e$

9. For the given series, $a_0 = 0$, $a_n = (-1)^{n-1}/n$, and $b_n = 0$. Parseval's identity states

$$\int_{-\pi}^{\pi} \ln^2(2\cos(x/2))\,dx = 2\pi a_0^2 + \pi\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \sum_{n=1}^{\infty}\frac{\pi}{n^2} = \frac{\pi^3}{6}$$

10. Note that $\Delta(f-g) = \Delta f - \Delta g = 0$, so $f - g$ is harmonic. Since $f = g$ on ∂B, $f - g$ is also zero on ∂B. The given theorem implies that $f - g = 0$ everywhere inside B, which is what we wanted.

11. Set $x = \pi$ to get

$$y = u(\pi, y) = \sum_{n=1}^{\infty} L_n \sinh(n\pi)\sin(ny).$$

Let m be a positive integer; then

$$\int_{-\pi}^{\pi} \sin(ny)\sin(my)\,dy = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

Thus

$$\int_{-\pi}^{\pi} y\sin(my)\,dy = L_m \sinh(\pi m)\pi \qquad \Longrightarrow \qquad L_m = (-1)^{m+1}\frac{2}{m\sinh(\pi m)}$$

12. Rearrange and identify the geometric series.

$$\int_0^{\infty} \frac{x}{e^x - 1}\,dx = \int_0^{\infty} \frac{xe^{-x}}{1 - e^{-x}}\,dx = \int_0^{\infty}\sum_{n=1}^{\infty} xe^{-nx}\,dx.$$

On the interval $[0, \infty)$ each term xe^{-nx} is positive. The monotone convergence theorem lets us write

$$\int_0^{\infty} \frac{x}{e^x - 1}\,dx = \sum_{n=1}^{\infty}\int_0^{\infty} xe^{-nx}\,dx = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$