Math 5C Spring 2010
Practice Exam 2 Solutions

M. Choice	
F. Resp. 1	
F. Resp. 2	
F. Resp. 3	
F. Resp. 4	
Total	

Name______________________________

Perm No.____________________________

Directions:

1. There are 125 points on this exam; 100 points = 100%.
2. Each multiple choice problem is 5 points.
3. Each multiple choice problem has exactly one best answer.
4. No multiple choice problem requires heavy computation.
5. Each free response problem is 20 points.
6. Free response questions require justification; no work, no credit.
7. **A blank free-response problem is awarded 5 points.**
8. No notes, books, or electronic devices are allowed.

Multiple Choice

1. (E) Since $\sum a_n$ converges, $a_n \to 0$. But then $1/a_n \not\to 0$, so $\sum 1/a_n$ can't converge.
2. (C)
3. (D) The limit comparison test. Note that answer E is nonsense—unless you're using the Lebesgue integral...
4. (B) By Green's theorem, the given integral equals $\iint 3\,dA = 3\text{area}(R)$.
5. (C) The first two terms are 1 and r, with $2(1+r) = 1/(1-r)$, so $r = 1/\sqrt{2}$.
6. (D)
7. (E) Only (A), (C), and (E) have terms which converge to zero. Of these, (A) and (C) converge absolutely.
8. (A)
9. (B) If the vote comes down to D and E, E will throw D overboard and take all the money no matter what. So if C were distributing coins, giving D a single coin will make D vote for C's proposal. That is, if the vote comes to pirate C, he'll give himself 99 coins, give D one coin, and give E nothing. Therefore when B is distributing coins C will vote against practically any proposal, but E will support B if he gets a single coin. This is not enough support for B to win, so B must also give D two coins to win the vote. The distribution for B is 97, 0, 2, 1. When A is distributing coins B will vote against nearly everything, but C will be happy if he gets a single coin. A needs the support of someone else, so he must give two coins to E, leaving 97 for pirate A.

Free Response

1. The appropriate integration by parts formula is

$$\iiint_R \nabla f \cdot \mathbf{G}\, dV = \iint_{\partial R} f\mathbf{G} \cdot d\mathbf{A} - \iiint_R f(\nabla \cdot \mathbf{G})\, dV.$$

On ∂R, $f = 0$. Also, $\nabla \cdot \mathbf{G} = 1$, so we have

$$\begin{aligned}\iiint_R \nabla f \cdot \mathbf{G}\, dV &= -\iiint_R (1 - x^2 - y^2 - z^2)\, dV \\ &= \int_0^{2\pi} \int_0^{\pi} \int_0^1 (\rho^2 - 1)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\ &= -\frac{8\pi}{15}\end{aligned}$$

2. By Green's theorem,

$$\text{area}(R) = \frac{1}{2} \oint_{\partial R} -y\, dx + x\, dy = \frac{1}{2} \int_0^{\infty} \frac{-9t^2 + 18t^5}{(1+t^3)^3}\, dt + \frac{18t^2 - 9t^5}{(1+t^3)^3}\, dt = \frac{1}{2} \int_0^{\infty} \frac{9t^2}{(1+t^3)^2}\, dt$$

This integral is easy; let $u = t^3$ to get

$$\text{area}(R) = \frac{3}{2} \int_0^{\infty} \frac{du}{(1+u)^2} = \frac{3}{2}.$$

3. Note that

$$\lim_{n\to\infty} \frac{(1/n)\ln(1 + 1/n)}{1/n^2} = \lim_{n\to\infty} n \ln\left(1 + \frac{1}{n}\right) = \ln \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = 1.$$

By limit comparison with the convergent series $\sum 1/n^2$, the series converges.

4. The series converges (absolutely) by the ratio test:

$$\lim_{n\to\infty} \left| \frac{(n+1)r^{n+1}}{nr^n} \right| = \lim_{n\to\infty} |r| \frac{n+1}{n} = |r| < 1.$$

Calling the sum S, notice that

$$\begin{aligned} S &= r + 2r^2 + 3r^3 + 4r^4 + \cdots \\ rS &= \phantom{r + {}} r^2 + 2r^3 + 3r^4 + \cdots \end{aligned}$$

Subtract the equations to get

$$(1-r)S = r + r^2 + r^3 + r^4 + \cdots = \frac{r}{1-r},$$

so $S = r/(1-r)^2$.