Math 5C Spring 2010
Practice Exam 3 Solutions

M. Choice	
F. Resp. 1	
F. Resp. 2	
F. Resp. 3	
F. Resp. 4	
Total	

Name______________________________

Perm No.______________________________

Directions:

1. There are 125 points on this exam; 100 points = 100%.
2. Each multiple choice problem is 5 points.
3. Each multiple choice problem has exactly one best answer.
4. No multiple choice problem requires heavy computation.
5. Each free response problem is 20 points.
6. Free response questions require justification; no work, no credit.
7. **A blank free-response problem is awarded 5 points.**
8. No notes, books, or electronic devices are allowed.

Multiple Choice

1. (E) $\|\sin x\|^2 = \int_{-\pi}^{\pi} \sin^2 x \, dx = \pi$
2. (A) Since f is even, all sin coefficients are zero.
3. (B)
4. (B) The function must be constant.
5. (B)
6. (A)
7. (A)
8. (E)
9. (A) The average of the left and right hand limits of f at π

Free Response

1. Note that for large n

$$\begin{aligned}\ln\left(1+\frac{1}{n}\right)^{n+1/2} &= \left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{n}\right)\\ &= \left(n+\frac{1}{2}\right)\left(\frac{1}{n}-\frac{1}{2n^2}++\frac{1}{3n^3}+O(n^{-4})\right)\\ &= 1+\frac{1}{12n^2}+O(n^{-3}).\end{aligned}$$

For small x, we have $e^x = 1+x+x^2/2+O(x^3)$. Together, this gives

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{n+1/2} &= \exp\left(1+\frac{1}{12n^2}+O(n^{-3})\right)\\ &= e\cdot\exp\left(\frac{1}{12n^2}+O(n^{-3})\right)\\ &= e\left[1+\left(\frac{1}{12n^2}+O(n^{-3})\right)+\frac{1}{2}\left(\frac{1}{12n^2}+O(n^{-3})\right)^2+O(n^{-3})\right]\\ &= e+\frac{e}{12n^2}+O(n^{-3})\end{aligned}$$

Finally,

$$\lim_{n\to\infty} n^2\left(\left(1+\frac{1}{n}\right)^{n+1/2}-e\right) = \lim_{n\to\infty}\left(\frac{e}{12}+O(n^{-1})\right) = \frac{e}{12}.$$

2. Multiply the given equation by x and take a derivative:

$$\sum_{n=0}^{\infty}\binom{2n}{n}x^n = \frac{d}{dx}\left(\frac{1-\sqrt{1-4x}}{2}\right) = \frac{1}{\sqrt{1-4x}}$$

Set $x = 1/16$ to get $2/\sqrt{3}$.

3. Since $a_0 = \pi/2$, Parseval's identity gives

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(\pi-|x|)^2\,dx = \frac{\pi^2}{2}+\frac{16}{\pi^2}\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right)$$

So the sum is

$$1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots = -\frac{\pi^4}{32}+\frac{\pi}{16}\int_{-\pi}^{\pi}(\pi-|x|)^2\,dx = \frac{\pi^4}{96}.$$

4. By the mean-value property for harmonic functions,

$$f(0,0,0) = \frac{1}{4\pi}\iint_S f\,d\sigma = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sin^2\phi\,d\phi\,d\theta = \frac{\pi}{4}.$$